LECTURE 10

Fields and Ordered Fields

Recall that an *ordered field* is a F together with a binary relation $\preceq$ satisfying the following axioms:

O1. Given $a, b \in F$, then either $a \preceq b$ or $b \preceq a$.
O2. If $a \preceq b$ and $b \preceq a$, then $a = b$.
O3. If $a \preceq b$ and $b \preceq c$, then $a \preceq c$.
O4. If $a \preceq b$, then $a + c \preceq b + c$.
O5. If $a \preceq b$ and $0 \preceq c$, then $ac \preceq bc$.

In the theorem below we use the notation $a \prec b$ to mean $a \preceq b$ and $a \neq b$.

THEOREM 10.1. *If F is an ordered field is an ordered field and $a, b, c \in F$, then:*

(i) *If $a \preceq b$, then $-b \preceq -a$.*
(ii) *If $a \preceq b$ and $c \preceq 0$, then $bc \preceq ac$.*
(iii) *If $0 \preceq a$ and $0 \preceq b$, then $0 \preceq ab$.*
(iv) *$0 \preceq a^2$ for all $a \in F$.*
(v) *$0 \prec 1$.*
(vi) *if $0 \prec a$, then $0 \prec a^{-1}$.*
(vii) *if $0 \prec a \prec b$, then $0 \prec b^{-1} \prec a^{-1}$.*

Taking $F = \mathbb{Q}$ and $\preceq$ to coincide with the the usual numerical inequality $\leq$, the above properties should seem quite elementary. However, in our context (the *axiomatic development* of number fields) these statements are properties which must first be proved before they can be employed.

Proof.

(i) Suppose that $a \preceq b$. Then by O4 we have

$$a + ((-a) + (-b)) \preceq b + ((-a) + (-b))$$

or

$$-b \preceq -a \quad .$$

(ii) If $c \preceq 0$, then

$$0 = c + (-c) \preceq 0 - c = -c$$

by (i). So $0 \preceq -c$. If $a \preceq b$, then by O5 we have

$$a(-c) \preceq b(-c) \quad ,$$

or

$$-ac \preceq -bc \quad .$$

Again by (i) we have

$$bc \preceq ac \quad .$$

(iii) If we put $a = 0$ and apply Axiom O5 we obtain

$$0 \preceq b \text{ and} 0 \preceq c \quad \Rightarrow \quad 0 \preceq bc \quad .$$

Except for the notation, this precisely statement (iii).

(iv) For any $a \in F$, either $a \preceq 0$ or $0 \preceq a$ by O1. If $0 \preceq a$, then $0 \preceq a^2$ by (iii).
If $a \preceq 0$, then $0 \preceq -a$ by (i) and so $0 \preceq (-a)^2$ by (iii). But by statement (iv) of Theorem 3.1, $(-a)(-a) = a^2$, hence $0 \preceq a^2$.

(v) Well, $0 \prec 1$ means that $0 \preceq 1$ and $0 \neq 1$. Now $0 \neq 1$ is certainly true; since otherwise Axiom M3 and statement (ii) of Theorem 3.1 would contradict each other. (We assume the field F does consists of more than one element.) To see that $0 \preceq 1$, we simply apply Axiom M3 and statement (iv) above to the case when $a = 1$.

$$0 \preceq 1^2 = 1 \cdot 1 = 1 \quad .$$

(vi) Suppose that $0 \prec a$; i.e $0 \preceq a$ and $0 \neq a$. If $a^{-1} \preceq 0$ then by (i) we have $0 \preceq (-a^{-1})$, and so by (iii)

$$0 \preceq a(-a^{-1}) = -1 \quad .$$

But then (i) implies

$$1 \preceq -0 = 0$$

which contradicts (v). Hence we cannot have $a^{-1} \preceq 0$; consequently $0 \prec a^{-1}$.

(vii) Assume $0 \prec a \prec b$. In view of (vi) we need only show that $b^{-1} \prec a^{-1}$. But since $a \neq b$, we know $a^{-1} \neq b^{-1}$; for if $a^{-1} = b^{-1}$, then $1 = a^{-1}a = b^{-1}a$, so $b = bb^{-1}a = a$. So we really only need to show $b^{-1} \preceq a^{-1}$.
Since $0 \prec a \prec b$, we have, in particular, $0 \preceq a$ and $0 \preceq b$, hence $0 \preceq ab$ by (iii). In fact $0 \prec ab$ by Statement (vi) of Theorem 3.1. But then $0 \prec (ab)^{-1}$ by (vi). We now apply Axiom 05 with $c = (ab)^{-1}$.

$$a(ab)^{-1} \preceq b(ab)^{-1}$$

The statement now follows from the identity

$$(ab)^{-1} = a^{-1}b^{-1} \quad .$$

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