LECTURE 8

Cardinality

In this lecture we shall discuss the relative *size* of sets. If S and T are sets with only a finite number of elements then this notion of relative size is straight-forward, the set S is the same *size* as the set T if it has the same number of elements and S is *larger* than the set T if S has more elements than T. If, however, S and T are sets with infinite numbers of elements then the notions of equivalent and relative sizes are a bit subtler.

We begin by stating a definition that tells us when two sets (infinite or finite) are the same *size*.

DEFINITION 8.1. *Two sets S and T are* ***equinumerous*** *if there exists a bijection from S to T.*

THEOREM 8.2. *Let $\mathcal{F}$ denote any family of sets, The relation*

$$S \sim T \quad \Longleftrightarrow \quad S \text{ and } T \text{ are equinumerous}$$

is an equivalence relation on $\mathcal{F}$.

The proof of this is easy, and will be assigned as homework.

We shall use the notation $S \sim T$ to indicate that S and T are equinumerous. Since this is an equivalence relation on any family $\mathcal{F}$ of sets, it partitions $\mathcal{F}$ up into disjoint equivalence classes. With each of these equivalence classes we shall associate a *cardinal number* indicating the size of the sets in that equivalence classs. This is done as follows.

NOTATION 8.3. For $n \in \mathbb{N}$, Let I_n denote the set of integers between 1 and n:

$$I_n = \{1, 2, 3, \dots, n\}$$

DEFINITION 8.4. *A set S is said to be* ***finite*** *if $S = \{\}$ or if there exists a natural number n, and a bijection $f : I_n \to S$. If S is finite, we say that the* ***cardinal number*** *(or* ***cardinality****) of S is 0 if $S = \{\}$, or n if the bijection is from I_n to S.If no such bijection exists we say that S is* ***infinite****,and that the cardinal number of S is* ***transfinite.***

DEFINITION 8.5. *A set S is* ***denumerable*** *if there exists a bijection $f : \mathbb{N} \to S$.*

DEFINITION 8.6. *If a set is finite or denumerable, then we say that the set is* ***countable.***

REMARK 8.7. If a set S is countable, then there either exists a bijection from I_n to S or there exists a bijection from $\mathbb{N}$ to S. Such a bijection (in either case) allows us to create a listing of the elements of S:

$$S = \{f(1), f(2), f(3), \dots\}$$

NOTATION 8.8. The cardinal number of a denumberable set is denoted by $\aleph_0$

EXAMPLE 8.9. Show that the cardinality of the set of integers is $\aleph_0$.

As usual we denote by $\mathbb{Z}$, the the set of integers, and by $\mathbb{N}$, the set of natural numbers . Then it is easy to verify that the map

$$f : \mathbb{N} \to \mathbb{Z} \quad n \mapsto \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

is a bijection. Thus,. $\mathbb{Z} \sim \mathbb{N}$, and so the cardinality of $\mathbb{Z}$ is $\aleph_0$.

LEMMA 8.10. *If T is an infinite subset of a set S, then S is also infinite.*

Proof. (Proof by Contradiction). Suppose that T is an infinite subset of a finite set S. Then, since S is finite, for some $n \in \mathbb{N}$, there is a bijection $f : I_n \to S$. Let $\{f(1), \dots, f(n)\}$ be the corrresponding listing of elements of S. Now remove from this list those elements of S that are not in T (this is a finite list so this is a finite procedure). What is left is a listing of elements of T which is equivalent to giving a bijection from some $I_m \to T$ with $m \leq n$. However, T is supposed to be infinite, so no such bijection is possible. We conclude that S must also be infinite.

THEOREM 8.11. *Let S be a countable set and let T be a subset of S. Then T is countable.*

Proof. If T is finite, then we are done. Suppose then that T is infinite. The preceding lemma implies that S is infinite, so it is denumerable (since it is infinite and countable). Hence, there is a bijection $f : \mathbb{N} \to S$. We can then list the elements of S as

$$S = \{f(1), f(2), \dots\}$$

Now define

$$A = \{n \in \mathbb{N} \mid f(n) \in T\}$$

This set is nonempty (since T is now assumed to be infinite) and so by the Well-Ordering Axiom for $\mathbb{N}$, it has a least element, say m_1. Similarly, the set $A \backslash \{m_1\}$ has a least member; call it m_2. Proceeding like this, we denote by m_k the least element of $A \backslash \{a_1, \dots, a_{k-1}\}$.

We now define a function $g : \mathbb{N} \to \mathbb{N}$ by $g(n) = a_n$, since T is infinite $g(n)$ is defined for every $n \in \mathbb{N}$. Since $a_{n+1} > a_n > \dots > a_1$ g must be injective. The composition $f \circ g : \mathbb{N} \to T$ is also then injective, and since every element of T must appear somewhere in this listing of S, $f \circ g$ is also surjective. Hence, there is a bijection from $\mathbb{N}$ to T so T is denumerable, hence countable.

THEOREM 8.12. *Let S be a non-empty set. Then the following conditions are equivalent.*

1. *S is countable*
2. *There exists a injective function $f : S \to \mathbb{N}$.*
3. *There exists a surjective function $g : \mathbb{N} \to S$.*

Proof.

$1 \Rightarrow 2$: Suppose S is countable. Then either S is finite or S is denumerable.

If S is denumerable, then by definition there exists a bijection $f : \mathbb{N} \to S$. But a bijection always has a bijective inverse and so f^{-1} will be, in particular, an injective function from S to $\mathbb{N}$.

If S is finite, then by definition there exists a bijection $f : I_n \to S$. The inverse $f^{-1} : S \to I_n$ of this mapping then exists and is bijective. Let $\iota : I_n \to \mathbb{N}$ be the natural *inclusion mapping*, where $i(k) = k$ for all $k \in I_n$. This is an injective function. The composition $i \circ f^{-1} : S \to I_n \to \mathbb{N}$, as the composition of two injective mappings, is itself injective.

$2 \Rightarrow 3$: Suppose $f : S \to \mathbb{N}$ is injective. Then $\tilde{f} : S \to f(S) \quad , \quad s \mapsto f(s)$ is bijective, and so $\tilde{f}^{-1} : f(S) \to S$ exists and is bijective. Then k be any element of $f(S)$ and define $g : \mathbb{N} \to S$ by

$$g(n) = \begin{cases} \tilde{f}^{-1}(n) & \text{if } n \in f(S) \\ k & \text{if } n \notin f(S) \end{cases}$$

This function is obviously surjective and so we are done.

$3 \Rightarrow 1$: Suppose there is a surjective function $g : \mathbb{N} \to S$. Then for each $s \in S$, $f^{-1}(\{s\})$ is a non-empty subset of $\mathbb{N}$. By the Well-Ordering Axiom each of these subsets has a minimal element, call it n_s. This gives us a one-to-one pairing of elements of S with elements of $\mathbb{N}$; hence, an injective function from S to $\mathbb{N}$.

EXAMPLE 8.13. Show that the set $\mathbb{Q}$ of rational numbers is countable

In view of the preceding theorem, it suffices to display an injective function from $\mathbb{Q}$ to $\mathbb{N}$. Every rational number r has a unique presentation as the ratio p/q of two integers which have no common factors. But then it is easy to check that the function

$$g : \mathbb{Q} \to \mathbb{N} \quad ; \quad \frac{p}{q} \to 2^p 3^q$$

is injective. Hence, $\mathbb{Q}$ is countable.

THEOREM 8.14. *The set of real numbers is not countable.*

Proof. By a preceding theorem, every subset of a countable set is countable, therefore it suffices to show that the subset of $\mathbb{R}$ consisting of real numbers between 0 and 1 is not countable. So set

$$J = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$$

If J is countable then we can form a list of its elements

$$J = \{x_1, x_2, \dots\}$$

Every real number is representable in terms of a (possibly infinite) decimal expansion, so we can write

$$\begin{aligned} x_1 &= 0.a_{11}a_{12}a_{13}\dots \\ x_2 &= 0.a_{21}a_{22}a_{33} \\ &\vdots \end{aligned}$$

Now we construct a real number $y \in J$ by defining

$$y = 0.b_1 b_2 b_3 \dots$$

where

$$b_i = \begin{cases} 2 & \text{if } a_{ii} \neq 2 \\ 3 & \text{if } a_{ii} = 2 \end{cases}$$

Now y evidently belongs to J, however, y can not be any of the x_n.For suppose

$$y = x_n$$

Then, by construction the n^{th} digit its decimal expansion can not be the same as that of x_n, so $y \neq x_n$.We conclude if we can not list the elements of J without running into a contradiction. Therefore, J and hence $\mathbb{R}$ is not countable.

NOTATION 8.15. Let S be a set. We shall denote the cardinal number of S by $|S|$. We shall say that $|S| = |T|$ if there exists a bijection $f : S \to T$ and that $|S| \leq |T|$ if there exists an injection $f : S \to T$ and we shall write $|S| < |T|$ if $|S| \leq |T|$ but $|S| \neq |T|$.

THEOREM 8.16. *Let S, T, and U be sets.*

1. *If $S \subset T$, then $|S| \leq |T|$.*
2. $|S| \leq |S|$
3. *If $|S| \leq |T|$ and $|T| \leq |U|$, then $|S| \leq |U|$.*
4. *If $m, n \in \mathbb{N}$ and $m \leq n$, then $|\{1, 2, 3, \dots, m\}| \leq |\{1, 2, 3, \dots, n\}|$*
5. *If S is finite, then $|S| \leq \aleph_0$*

DEFINITION 8.17. *Given any set S, we denote the collection of all subsets of S, by $\mathcal{P}(S)$ and refer to it as the **power set** of S.*

THEOREM 8.18. *For any set S,we have $|S| < |\mathcal{P}(S)|$*

Proof. If S and $\mathcal{P}(S)$ are equinumerous, then there must exist a bijection $f : S \to \mathcal{P}(S)$. The natural inclusion map

$$i : S \to \mathcal{P}(S) \quad , \quad s \mapsto \{s\}$$

is obviously injective, so $|S| \leq |\mathcal{P}(S)|$., We need to show that no map from S to $\mathcal{P}(S)$ can be surjective.

Let f be an injective map from S to $\mathcal{P}(S)$. Then for each $x \in S$, each $f(x)$ is a subset of S. Now for some x in S it may be that x is in the subset $f(x)$, while for others may not be. Let

$$T = \{x \in S \mid x \notin f(x)\}$$

Now $T \subset S$, so $T \in \mathcal{P}(S)$. If f is surjective then $T = f(y)$ for some $y \in S$. But now either $y \in T$ or $y \notin T$; however, both possibililities lead to contradictions.

- If $y \in T$, then $s \notin f(y) = T$.
- If $y \notin T$,then $y \notin f(y) \quad \Rightarrow \quad y \in T$.

THEOREM 8.19. $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$

PROBLEM 8.1. *Is there a set S such that $|\mathbb{N}| < |S| < |\mathbb{R}|$?*