LECTURE 7

Functions

1. Functions as Special Cases of Relations

DEFINITION 7.1. *Let A and B be sets. A* ***function*** *between A and B isa non-empty relation $f \subseteq A \times B$ such that*

$$(a,b) \in f \text{ and } (a,b') \in f \quad \Rightarrow \quad b = b'$$

The ***domain*** *of f is the set of all first elements of f:*

$$domain(f) = \{a \in A \mid (a,b) \in f\}$$

and the ***range*** *of f is the set of all second elements of f:*

$$range(f) = \{b \in B \mid (a,b) \in f\}$$

If it happens that the domain of f is the entire set A we say that f is a ***function from*** *A* ***to*** *B, and write*

$$f : A \to B$$

DEFINITION 7.2. *A function $f : A \to B$ is called* ***surjective*** *if $B = range(f)$.*

DEFINITION 7.3. *A function $f : A \to B$ is called* ***injective*** *(or* ***one-to-one****) if, for all $\forall a, a' \in A$*

$$f(a) = f(a') \quad \Rightarrow \quad a = a'$$

NOTATION 7.4. Suppose that f is a function from A to B. If C is a subset of A, we denote by $f(C)$ the subset

$$f(C) = \{b \in B \mid b = f(c) \text{ for some } c \in C\}$$

The set $f(C)$ is called the image of C in B by f. If D is a subset of B, the set

$$f^{-1}(A) \equiv \{a \in A \mid f(a) \in D\}$$

is called the **pre-image** of D in A by f.

THEOREM 7.5. *Suppose that $f : A \to B$. Let C, C_1, and C_2 be subsets of A, and let D, D_1 and D_2 be subsets of B; then*

1. $C \subseteq f^{-1}(f(C))$
2. $f\left(f^{-1}(D)\right) \subseteq D$
3. $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$
4. $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$
5. $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$
6. $f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$
7. $f^{-1}(B \backslash D) = A \backslash f^{-1}(D)$

Proof:

1. Let c be an arbitary element of C, then $f(c)$ is an element of $f(C)$ since $c \in C$. But then c is an element of $f^{-1}(f(C))$,because this set consists of all points in C that land on some element $d \in f(C)$. So every element of C is an element of $f^{-1}(f(C))$; i.e., $C \subseteq f^{-1}(f(C))$

2. Let d be an arbitrary element of D. Then d lies in either $D \cap f(A)$ or $f^{-1}(\{d\}) = \{\}$. In the first case, we obviously have $d \in f\left(f^{-1}(\{d\})\right) = \{d\} \subset D$. In second case, we have $f\left(f^{-1}(\{d\})\right) = f(\{\}) = \{\} \subset D$. So in either case $f\left(f^{-1}\{d\}\right) \subset D$. Hence, $f\left(f^{-1}(D)\right) \subseteq D$.
3. Homework.
4. Homework.
5. Homework.
6. Homework.
7. Homework.

THEOREM 7.6. *Suppose f is a funciton from A to B. Let Let C, C_1, and C_2 be subsets of A, and let D be a subset of B; then*

1. *If f is injective, then $f^{-1}(f(C)) = C$.*
2. *If f is surjective, then $f\left(f^{-1}(D)\right) = D$.*
3. *If f is injective, then $f(C_1 \cap C_2) = f(C_1) \cap f(C_2)$.*

Proof:

1. Homework.
2. Homework.
3. From the preceding Theorem we know for any map $f : A \to B$, and any subsets C_1, C_2 of A, we have
$$f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$$
we need to show that
$$f(C_1) \cap f(C_2) \subseteq f(C_1 \cap C_2)$$
if f is injective. Suppose $y \in f(C_1) \cap f(C_2)$. Then there exists $c_1 \in C_1$ and $c_2 \in C_2$ such that $y = f(c_1)$ and $y = f(c_2)$. But then since f is injective
$$\begin{aligned} f(c_1) &= f(c_2) \quad \Rightarrow \quad c_1 = c_2 \\ &\Rightarrow \quad c_1 \in C_2 \\ &\Rightarrow \quad c_1 \in C_1 \cap C_2 \\ &\Rightarrow \quad y = f(c_1) \in f(C_1 \cap C_2) \end{aligned}$$

2. Composition of Functions

Let $f : A \to B$ and $g : B \to C$ be two functions such that the range of f coincides with the domain of g. Then the **composite** of f and g is the function $g \circ f : A \to C$ whose rule is
$$g \circ f(x) = g(f(x)) \quad , \quad \forall\, x \in A \quad .$$

In terms of ordered pairs we have the following defintion:

DEFINITION 7.7. *If $f : A \to B$ and $g : B \to C$ then the composition of f and g is the function $g \circ f : A \to C$ defined by*
$$\{(a, c) \in A \times C \mid \exists b \in B \ s.t. \ (a, b) \in f \ and \ (b, c) \in g\}$$

EXAMPLE 7.8. Let $\mathbb{E}$ denote the set of even integers; $\mathbb{Z}$, the set of integers; and $\mathbb{N}$ the set of natural numbers. Let $f : \mathbb{E} \to \mathbb{Z}$ be the map defined by
$$f(e) = \frac{e}{2} \quad , \quad \forall\, e \in \mathbb{E}.$$
Let $g : \mathbb{Z} \to \mathbb{N}$ be the map defined by
$$g(z) = z^2 \quad , \quad \forall\, z \in \mathbb{Z} \quad .$$

Then the composite mapping $g \circ f : \mathbb{E} \to \mathbb{N}$ has the rule

$$e \;\mapsto\; (g \circ f)(e) = \left(\frac{e}{2}\right)^2 = \frac{e^2}{4} \quad .$$

Note that the composite function with the opposite order $f \circ g$ is *not defined*, since the domain $\mathbb{E}$ of f is not contained in the range $\mathbb{N}$ of g.

EXAMPLE 7.9. Let

$$\begin{aligned} f &: \mathbb{Z} \to \mathbb{Z} \quad ; \quad f(n) = n - 1 \\ g &: \mathbb{Z} \to \mathbb{Z} \quad ; \quad g(n) = n^2 \end{aligned}$$

Then

$$\begin{aligned} (f \circ g)(n) &= f(g(n)) \\ &= f\left(n^2\right) \\ &= n^2 + 1 \quad , \end{aligned}$$

while

$$\begin{aligned} (g \circ f)(n) &= g(f(n)) \\ &= g(n-1) \\ &= (n-1)^2 \\ &= n^2 - 2n + 1 \quad . \end{aligned}$$

So even though both $f \circ g$ and $g \circ f$ are both well-defined, $f \circ g$ **is not the same function as** $g \circ f$.

We conclude that **the composite of two functions depends on the order in which they are composed**.

DEFINITION 7.10. *Let $f : A \to B$ be a bijection. The **inverse function** of f is the function $f^{-1} : B \to A$ defined by*

$$f^{-1} = \{(b, a) \in B \times A \mid b = f(a)\}$$

Let us first verify that the inverse of a bijection f is indeed a function. In order for a subset of $B \times A$ to be a function we must have

$$f^{-1}(b) = f^{-1}(b') \quad \Rightarrow \quad b = b'$$

But since f is surjective, there exists $a, a' \in A$ such that $f(a) = b$ and $f(a') = b'$. And so the "function", f^{-1} will be defined on all of B. However, since f is injective, the sets $f^{-1}(\{b\})$ and $f^{-1}(\{b'\})$ must contain only a single element of A. Hence $f^{-1}(\{b\}) = \{a\}$ and $f^{-1}(\{b'\}) = \{a'\}$. And now

$$\begin{aligned} f^{-1}(b) = f^{-1}(b') &\Rightarrow \{a\} = \{a'\} \\ &\Rightarrow a = a' \\ &\Rightarrow f(a) = f(a') \\ &\Rightarrow b = b' \end{aligned}$$

DEFINITION 7.11. *If A is a set then the identity function on A is the function i_A defined by*

$$i_A = \{(a, a') \in A \times A \mid a' = a\}$$

THEOREM 7.12. *If $f : A \to B$ is a bijection, then*

1. *$f^{-1} : B \to A$ is a bijection*
2. *$f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$*

THEOREM 7.13. *Let $f : A \to B$ and $g : B \to C$ be bijections. Then the composition $g \circ f : A \to C$ is a bijection and $(gof)^{-1} = f^{-1} \circ g^{-1}$.*

Proof. We know from a preceding theorem that $g \circ f$ is bijection. Thus, $g \circ f$ has an inverse. Now

$$g \circ f = \{(a, c) \in A \times C \mid \exists b \in B \text{ such that } (a, b) \in f \text{ and } (b, c) \in g\}$$

so that

$$\begin{aligned} (g \circ f)^{-1} &= \{(c, a) \in C \times A \mid \exists b \in B \text{ such } (a, b) \in f \text{ and } (b, c) \in g\} \\ &= \left\{(c, a) \in C \times A \mid \exists b \in B \text{ such that } (b, a) \in f^{-1} \text{ and } (c, b) \in g^{-1}\right\} \\ &= f^{-1} \circ g^{-1} \end{aligned}$$