LECTURE 6

Relations

1. Ordered Pairs

Recall that when we specify a set by listing its elements, as in

$$S = \{a, b, c\} \quad ,$$

the order in which the elements appear is immaterial:

$$S = \{b, a, c\} = \{c, a, b\} = \{b, c, a\} = \dots$$

Sometimes, however, we require a listing of elements in which order is important. For example, the vector $[1, 2, 3]$ is distinct from the vector $[2, 1, 3]$. There are several ways of introducing ordered sets into our axiomatics (see Section 2.6 of the text for one example), however, the following definition will suffice for our purposes

Definition 6.1. *An **ordered pair** set is a set S with 2 elements, together with a 1-to-1 map $f : \{1, 2\} \to S$.*

Notation 6.2. We shall use parentheses () instead of braces {} to list the elements of a ordered pair. Thus, we shall denote the ordered pair defined by

$$\{a, b\}$$
$$f \quad : \quad \left\{ \begin{array}{l} 1 \to a \\ 2 \to b \end{array} \right.$$

by

$$(a, b)$$

is the ordered pair whose first element (the element what is the image of 1 by f) is a and whose second element (the element that is the image of 2 by f).

Definition 6.3. *If A and B are sets, then the **Cartesian product** of A and B, written $A \times B$, is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$.*

$$A \times B \equiv \{(a, b) \mid a \in A \text{ and } b \in B\}$$

2. Relations

Definition 6.4. *Let A and B be sets. A **relation** between A and B is any subset R of $A \times B$. If $a \in A$, $b \in B$, and $(a, b) \in R$ then we say that a and b are **related**.*

Definition 6.5. *A relation R on a set S (i.e a non-empty subset of the Cartesian product $S \times S$) is an **equivalence relation** if it has the following properties for all $x, y, z \in S$. .*

1. *$(x, x) \in R$ (the reflexive property)*
2. *If $(x, y) \in R$ then $(y, x) \in R$ (the symmetric property)*
3. *If $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$ (the transitive property)*

If x is an element of S and R is an equivalence relation on S then the ***equivalence class of*** *x is the set of all elements y of S that are related to x by R.*

NOTATION 6.6. If R is an equivalence relation on S, we shall use the notation $x \sim y$ to indicate that $(x, y) \in R$. Thus, three criteria above can be written

$$\begin{array}{rcll} x \sim x & & & \forall x \in S \\ x \sim y & \Rightarrow & y \sim x & \forall x, y \in S \\ (x \sim y \text{ and } y \sim z) & \Rightarrow & x \sim z & \forall x, y, z \in S \end{array}$$

If x is an element of a set S with an equivalence relation R, we signify by $\tilde{x}$ the equivalence class of x; i.e.,

$$\tilde{x} = \{y \in S \text{ such that } y \sim x\}$$

EXAMPLE 6.7. If $\mathbb{R}$ is the set of real numbers then the relation $x \sim y$ if $y = x$ is an equivalence relation on $\mathbb{R}$.

EXAMPLE 6.8. If $\mathbb{R}$ is the set of real numbers then the relation $x \sim y$ if $y \geq x$ is not an equivalence realtion. (Both the symmetric property and the transitive properties fail)

EXAMPLE 6.9. If $S = \mathbb{R} - \{0\}$is the set of real numbers exc, the relation

$$x \sim y \text{ if } x/y \text{ is a rational number}$$

is an equivalence relation on S.

EXAMPLE 6.10. If S is the set of points in the plane, then the relation

$$(x, y) \sim (x', y') \;\text{ if } x = x'$$

is an equivalence relation on S.

DEFINITION 6.11. *A* ***partition*** *of a set S is a collection $\mathcal{P}$ of nonempty subsets of S such that*

1. *Each $x \in S$ belongs to some $P \in \mathcal{P}$.*
2. *If $P, P' \in \mathcal{P}$ and $P \neq P'$ then $P \cap P' = \{\}$.*

We shall refer to the individual subsets of S that comprise a partition $\mathcal{P}$ of S as the ***pieces*** *of the partition.*

DEFINITION 6.12. *Let $\mathbb{Z}$ denote the set of all integers and set*

$$\begin{array}{rcll} \mathbb{E} & = & \{z \in \mathbb{Z} \mid z = 2k \text{ for some } k \in \mathbb{Z}\} & \text{(the set of even integers)} \\ \mathbb{O} & = & \{z \in \mathbb{Z} \mid z = 2k + 1 \text{ for some } k \in \mathbb{Z}\} & \text{(the set of odd integers)} \end{array}$$

Then $\mathcal{P} = \{\mathbb{E}, \mathbb{O}\}$ is a partiion of $\mathbb{Z}$.

THEOREM 6.13. *Let $\sim$ be an equivalence relation on a set S. Then*

$$\mathcal{P} = \{\tilde{x} \mid x \in S\}$$

is a partition of S. Conversely, if $\mathcal{P}$ is a partition of S then $\mathcal{P}$ defines an equivalence relation on S by

$$x \sim y \quad \text{if} \quad x \text{ and } y \text{ belong to the same piece of the partition } P$$

Proof.

$\Rightarrow$ Let $\mathcal{P}$ be the set of all equivalence classes in S. Then since every element of S is equivalent to itself (the reflexive property), every element of S belongs to some piece of $\mathcal{P}$. Furthermore, every element of S can only belong to one distince equivalence class. For suppose

$$z \in \tilde{x} \cap \tilde{y}$$

Let x' be an arbitrary element of $\tilde{x}$. Then the transitivity property of the equivalence relation implies

$$z \sim x \text{ and } x' \sim x \quad \Rightarrow \quad x' \sim z$$

But since z is also an element of $\tilde{y}$, $z \sim y$, and

$$\begin{aligned} x' &\sim z \text{ and } z \sim y \quad \Rightarrow \quad x' \sim y \\ &\Rightarrow \quad x' \in \tilde{y} \quad \forall x' \in \tilde{x} \\ &\Rightarrow \quad \tilde{x} \subset \tilde{y} \end{aligned}$$

Reversing the roles of $\tilde{x}$ and $\tilde{y}$ in the argument above, one can similary conclude that $\tilde{y}$ is a subset of $\tilde{x}$. But

$$\tilde{x} \subset \tilde{y} \text{ and } \tilde{y} \subset \tilde{x} \quad \Rightarrow \quad \tilde{x} = \tilde{y}$$

We conclude that if $\tilde{x}$ and $\tilde{y}$ contain a common element z then in fact $\tilde{x} = \tilde{y}$. Thus, every element of S belongs to an equivalence class, and any two equivalence classes are either the same or disjoint. Hence, $\mathcal{P} = \{\tilde{x} \mid x \in S\}$ is a partition of S.

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