LECTURE 3

Methods of Proof, Cont'd

Last time we discussed some of the basic techniques for proving propositions. We began with the notion of a **direct proof** and one of its implementations, the **forward-backward method**. Then we discussed an alternative to the direct proof, **proof by contradiction**.

I would like to begin today's lecture with another example of proof by contradiction.

EXAMPLE 3.1. Use the method of proof by contradiction to show that the equation

$$x^2 + 2mx + 2n = 0$$

has no odd roots if m and n are odd.

This proposition is equivalent to one of the form $P \Rightarrow Q$ if we take P to be the statement "m and n are odd integers", and Q to be the statement "there exists no odd integer x such that $x^2 + 2mx + 2n = 0$". Not-Q is then the statement that there exists an odd integer x such that $x^2 + 2mx + 2n = 0$.

To apply the method of proof by contradiction, we suppose P and not-Q are true and look for a contradiction with known facts.

So let x be an odd integer satisfying

$$x^2 + 2mx + 2n = 0$$

and suppose both m and n are odd. Then

$$x^2 = -2(mx + n) \quad .$$

The right hand side, being a multiple of 2, is clearly even. So x^2 is even. On the other hand, x is supposed to be odd. In your last homework assignment you proved (hopefully) that if x is odd, then x^2 is necessarily odd. We have thus arrived at a contradiction with a known fact. Since P and not-Q can not be simultaneously true, we have

$$\text{"}P \text{ is true"} \Rightarrow \text{"not-}Q \text{ is false"}$$

or

$$P \Rightarrow Q \quad .$$

1. The Contrapositive Method

The **contrapositive method** is a variation of the proof by contradiction method in which one tries to work forward from the hypothesis "not-Q is true" to conclude that "not-P" is true. For if we can show that

$$\text{not-Q} \Rightarrow \text{not-P}$$

then we can conclude that

$$P \Rightarrow Q \quad .$$

The justification for conclusion runs as follows:

$$\text{"}P \text{ is true"} \Rightarrow \text{"not-}P \text{ is false"} \qquad i$$

$$\text{“not-}P\text{ is false” and “not-}Q \Rightarrow \text{not-}P\text{ is true”} \Rightarrow \text{“not-}Q\text{ is false”} \qquad ii$$

$$\text{“not-}Q\text{ is false”} \Rightarrow \text{“}Q\text{ is true”} \qquad iii$$

EXAMPLE 3.2. Let's use the contrapositive method to prove

Proposition If n is an integer and n^2 is an odd integer, then n is odd.

proof.

In this proposition the hypothesis P is "n is an integer and n^2 is an odd integer" and the conclusion Q we're trying to reach is "n is odd". So not-Q is "n is even" and not-P is "n^2 is even". Thus, in view of the fact that

$$\text{“not-}Q \Rightarrow \text{not-}P\text{ is true”} \Rightarrow \text{“}P \Rightarrow Q''$$

it suffices to prove that if n is even, then n^2 is even. And this we have already done; so we're finished. ∎

2. Proof by Construction

Another method of proof is particularly useful for proving statements involving existential quantifiers (e.g., "there exists at least one ...").

This method works as follows:

In order to prove a statement of the form

"If *such and such*, then there exists an *object* such that *so and so*."

it suffices to construct (guess, produce, devise an algorithm to produce, etc.), using the hypothesis "*such and such*", the *object* in the conclusion and show that it satisfies the properties "*so and so*".

EXAMPLE 3.3. Prove that if $a < b$, then there exists a real number c such that

$$a < c < b \quad .$$

Proof. Set

$$c = \frac{a+b}{2} \quad .$$

Then

$$c - a = \frac{a+b-2a}{2} = \frac{b-a}{2} > 0$$

since $b > a$; and

$$b - c = \frac{2b-a-b}{2} = \frac{b-a}{2} > 0$$

for the same reason. Thus, we have constructed a number c with the desired properties.

EXAMPLE 3.4.

Proposition If a, b, c, d, e and f are real numbers such that

$$(ad - bc) \neq 0 \quad ,$$

then the two equations

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

can be solved for x and y.

proof. Our hypothesis is that a, b, c, d, e and f are real numbers such that

$$(ad - bc) \neq 0 \quad ,$$

and the conclusion we are trying to prove is that there exists real numbers x and y such that

$$ax + by = e \quad , \tag{3.1}$$

$$cx + dy = f \quad . \tag{3.2}$$

Now set

$$\begin{aligned} x' &= \frac{de - bf}{ad - bc} \quad , \\ y' &= \frac{af - ce}{ad - bc} \quad . \end{aligned}$$

These numbers are well defined because, by hypothesis, $ad - bc \neq 0$.

Now insert these expressions for x and y into (3.2). It is easily verifed that

$$\begin{aligned} ax' + by' &= e \\ cx' + dy' &= f \end{aligned}$$

and so we can satisfy the conclusion of the proposition by setting $x = x'$ and $y = y'$.