LECTURE 1

Proofs and Logic

The primary purpose of this course is to introduce you, most of whom are mathematics majors, to the most fundamental skills of a mathematician; the ability to read, write, and understand proofs. Indeed, this is a course where **proofs** matter more than results.

That said, I should also stress that this is *not* supposed to be a killer course. Yes, we are going to be rigorous and meticulous; but we will take our time to cover the material. And while we will be often dealing in abstractions; our purpose shall be to develop concrete ways of handling far reaching concepts.

1. Logic

In order to get our bearings, let us begin with a discussion of logic and proof. Much of this discussion will appear as common sense. However, not all common sense is logical, nor does every common sensical argument constitute a proof. For this reason, we must delineate from the start, exactly what constitutes a logical argument.

1.1. Statements.

DEFINITION 1.1. *A **statement** is a declarative sentence that is either true or false.*

Each of the following sentences is a statement:

Every square has four sides.
π is a rational number.
Orange is the best color.

Note that the first statement is true, the second is false and the third is merely an opinion.

Let us now lay out the means by which we manipulate statements in a **logical** manner.

1.2. Compound Statements.

Suppose P and Q are statements which are either true or false. Then there are several ways we can create new statements which are also either true or false.

1.2.1. *Logical Connectives.*

DEFINITION 1.2. ***If P and Q are statements, then "P and Q" is a true statement only if P and Q are both true; otherwise "P and Q" is false.***

Thus,

$$\left.\begin{array}{r}P \text{ is true and } Q \text{ is true.}\end{array}\right\} \quad \Rightarrow \quad \text{“}P \text{ and } Q\text{” is true}$$

$$\left.\begin{array}{r}P \text{ is true and } Q \text{ is false.}\\ Q \text{ is true and } P \text{ is false.}\\ Q \text{ is false and } P \text{ is false.}\end{array}\right\} \quad \Rightarrow \quad \text{“}P \text{ and } Q\text{” is false}$$

In standard English the conjunction "or" can be used in two distinct ways depending on the context: first of all it can be used to exclude one or the other of two possibilities:

The result of a coin toss is either head *or* tails.

We refer to this usage as the *exclusive or.* The word "or" can also be used to include two possibilities:

I need 6 or 7 dollars.

In mathematics, one always uses the logical connective "or" in the inclusive sense.

DEFINITION 1.3. ***If P and Q are statements, then "P or Q" is a true statement only if P, Q is true, or P and Q are both true; otherwise "P or Q" is false.***

So there are always three possibilities for "P or Q" to be a true mathematical statement and only one possibility for it to be false:

$$\left.\begin{array}{l} P \text{ is true and } Q \text{ is true.} \\ P \text{ is true and } Q \text{ is false.} \\ Q \text{ is true and } P \text{ is false.} \end{array}\right\} \quad \Rightarrow \quad \text{"}P \text{ or } Q\text{" is true}$$

$$P \text{ is false and } Q \text{ is false.}\} \quad \Rightarrow \quad \text{"}P \text{ or } Q\text{" is false}$$

1.2.2. ***Universal Quantifiers.*** The following statements contain universal quantifiers.

For all real numbers x, $x^2 \neq -1$.
All triangles have three sides.
For each real number a, $a^2 \geq 0$.

Notice that in each of the statements above, a property is attributed to **all** members of a set; this is what we mean by a universal quantifier.

NOTATION 1.4. As a short hand for the phrase "*for all*" we shall use the symbol $\forall$ (an up-side-down A).

1.2.3. ***Existential Quantifiers.*** The following statements contain existential quantifiers

Some integers are prime.
There exists a integer between 7.5 and 9.1.
The exists an irrational real number.

Notice that in each of these statements a property is attributed to **at least one** element of a set; this is what one means by a existential quantifier.

NOTATION 1.5. As a short hand for the phrase "there exists", we shall often use the symbol $\exists$ (a backwards E).

NOTATION 1.6. As a short hand for the phrase "such that", we shall often use the abbreviation "*s.t.*".

As an example of our notational short-hand we note that

$$\forall x, \quad \exists y \quad s.t. \quad y = x^2$$

translates as "for all x, there exists a y such that y equals x^2".

1.2.4. ***Negation.*** The **negation**, not-P, of a statement P is the statement such that not-P is true exactly when P is false, and not-P is false exactly when P is true.

In most cases you can transform a statement into its negation by inserting a "not" in the appropriate place.

$$\text{A is B.} \longrightarrow \text{A is not B.}$$

The negation of compound statements works as follows:

The negation of "P and Q" is "not-P or not-Q".
The negation of "P or Q" is "not-P and not-Q".

The negation of universal and existential quantifiers works as follows:

The negation of a statement with a universal quantifier is a statement with an existential quantifier.

The negation of a statement with an existential quantifier is a statement with a universal quantifier.

For example, the negation of the statement

"All crayons are blue$''$,

which has a universal quantifier is

"Not all crayons are blue"

which if true, would of course imply that at least one crayon was not blue; i.e. a statement with an existential quantifier.

In summary

A is B	$\xrightarrow{\text{negation}}$	A is not B
A and B	$\xrightarrow{\text{negation}}$	not-A or not-B
A or B	$\xrightarrow{\text{negation}}$	not-A and not-B
All A are B	$\xrightarrow{\text{negation}}$	at least one A is not-B
At least one A is B	$\xrightarrow{\text{negation}}$	all A are not-B

NOTATION 1.7. If P is a statement we shall sometimes employ the notation $\tilde{}P$ to indeicate its negation "not-P".

1.3. Conditional Statements. In mathematics one deals primarily with **conditional statements**; that is to say statements of the form

$$\text{If } P, \text{ then } Q.$$

which is written symbolically as

$$P \Rightarrow Q \quad .$$

Such a statement means that the truth of P guarantees the truth of Q. More explicitly

- $P \Rightarrow Q$ is true if Q is true whenever P is true.
- $P \Rightarrow Q$ is false if Q can be false when P is true.

The statement P is called the *hypothesis*, or *premise*, and the statement Q is called the *conclusion*. Here are some examples:

If x and y are integers, then $x+y$ is an integer.
$x \neq 0 \;\Rightarrow\; x^2 > 0$.

There are several ways of phrasing a conditional statement, all of which mean the same thing:

$$\left.\begin{array}{l}\text{If } P\text{, then } Q.\\ P \text{ implies } Q.\\ P \text{ is sufficient for } Q.\\ Q \text{ provided that } P.\\ Q \text{ whenever } P.\end{array}\right\} \quad \sim \quad P \Rightarrow Q$$

1.4. The Contrapositive of a Conditional Statement. The contrapositive of a conditional statement

If P, then Q.

is the conditional statement

If *not*-Q , then *not*-P

For example, the contrapositive of

If $x < 6$, then $x < 8$

is

If x is not less than 8, then x is not less than 6

or, equivalently,

If $x \geq 8$, then $x \geq 6$.

In this example, the truth of the original conditional statement seems to guarantee the truth of its contrapositive. In fact,

The conditional statement "$P \;\Rightarrow\; Q$" is equivalent to its contrapositive "not-$Q \;\Rightarrow$ not-P".

Let's prove

" $P \;\Rightarrow\; Q$" implies "not-$Q \;\Rightarrow$ not-P" .

By hypothesis, if P is true, then Q is true. Suppose not-Q is true. Then Q is false. But then P can not be true, since that would contradict our hypothesis. So not-P must be true.

EXAMPLE 1.8. Prove that

"not-$Q \;\Rightarrow\;$ not-P" implies "$P \;\Rightarrow\; Q$" .

1.5. The Converse of a Conditional Statement. The converse of the conditional statement

$$P \Rightarrow Q$$

is the conditional statement

$$Q \Rightarrow P \quad .$$

It is important to note that **the truth of a conditional statement does not imply the truth of its converse**. For example, it is true that

If x is an integer, then x is a real number;

but the converse of this statement

If x is a real number, then x is an integer,

is certainly not true.

However, there are some situations in which both a conditional statement and its converse are true. For example, both

If the integer x is even, then the integer $x + 1$ is odd

and its converse

If integer $x + 1$ is odd, then the integer x is even

are true. We can state this fact more succinctly by saying

The integer x is even **if and only if** the integer $x + 1$ is odd .

More generally, the statement

P if and only if Q

which may be abbreviated

P iff Q

or

$$P \Leftrightarrow Q$$

means

"$P \Rightarrow Q$" and "$Q \Rightarrow P$" .

"P if and only if Q" is called a **biconditional statement.** When "$P \Leftrightarrow Q$" is a true biconditional statement, P is true exactly when Q is true, and so the statements P and Q can be regarded as equivalent statements (when inserted in other statements).