Math 4023
Homework Set 7

1. Find an example of a sequence of real numbers satisfying each set of properties.

   (a) Cauchy, but not monotone.
   (b) Monotone, but not Cauchy.
   (c) Bounded, but not Cauchy.

2. Show that a decreasing sequence is convergent if and only if it is bounded.

3. Show that if (s_n) is an unbounded decreasing sequence, then $\lim s_n = -\infty$.

4. Prove that every Cauchy sequence is bounded.

5. Use the definition of the limit of a function to prove that $\lim_{x\to 2} x^3 = 8$.

6. Prove that the limit $\lim x \to 0^+ 1/x$ does not exist.

7. Use the definition of the limit of a function to prove that if $f : D \to \mathbb{R}$ and if c is an accumulation point of D, then f can have only one limit at c.

8. Let $f : D \to \mathbb{R}$ and let c be an accumulation point of f. Prove that the following statements are equivalent:

   (a) f does not have a limit at c.
   (b) There exists a sequence (s_n) in D with each $s_n \neq c$ such that (s_n) converges to c, but $(f(s_n))$ does not convergent in $\mathbb{R}$.

9. Let f. g and h be functions from D to $\mathbb{R}$ and let c be an accumulation point of D. Suppose that $f(x) \leq g(x) \leq h(x)$ for all $x \in D$ with $x \neq c$, and that

$$\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x)$$

Prove that

$$\lim_{x\to c} g(x) = L$$

10. Let $f : d \to \mathbb{R}$ and let c be an accumulation point of D. Suppose that f has a limit at c. Prove that f is bounded on a neighborhood of c (i.e,, prove that there exists a deleted neighborhood U of c and a real number m such that $|f(x)| \leq M$ for all $x \in U$).