Math 4023
Homework Set 4

1. What is the interior of the set $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$? What is its boundary? Does S have any accumulation points?

2. Suppose $x \in \mathbb{R}$ and $\varepsilon > 0$. What is the interior of the set $N^*(x,\varepsilon) \equiv \{y \in \mathbb{R} \mid |y - x| < \varepsilon, y \neq x\}$. What is the boundary of $N^*(x,\varepsilon)$? What are the accumulation points of $N^*(x,\varepsilon)$?

3. Prove that if A is open and B is closed, then $A \backslash B$ is open and $B \backslash A$ is closed. ($A \backslash B$ is the subset of A containing no elements of B.)

4. Prove that $cl(S) \backslash int(S) = bd(S)$.

5. Let S be an infinite bounded subset of $\mathbb{R}$ and let $x = \sup(S)$. Prove that x is an accumulation point of S.

6. Prove that if x is an accumulation point of $S \subseteq \mathbb{R}$, then every neighborhood of x contains infinitely many point of S.

7. Let S be a subset of $\mathbb{R}$. Prove the following statements

   (a) $bd(S) = cl(S) \cap cl\left(\mathbb{R} \backslash S\right)$
   (b) $bd(S)$ is a closed set.

8. Let S and T be subsets of $\mathbb{R}$. Prove the following statements.

   (a) $cl\left(cl(S)\right) = cl\left(S\right)$
   (b) $cl\left(S \cup T\right) = cl(S) \cup cl(T)$
   (c) $cl\left(S \cap T\right) \subseteq cl(S) \cap cl(T)$
   (d) Find an example to the the equality need not hold for part (c).

8. Let S and T be subsets of $\mathbb{R}$. Prove the following statements.

   (a) $int(S)$ is an open set.
   (b) $int\left(int(S)\right) = int\left(S\right)$
   (c) $int\left(S \cup T\right) \subseteq int(S) \cup int(T)$
   (d) $int\left(S \cap T\right) = cl(S) \cap int(T)$
   (e) Find an example to the the equality need not hold for part (c).