Math 4013
Solutions to Homework Problems from Chapter 7

Section 7.1

7.1.1. Evaluate $\int_\sigma f\,ds$ where $f(x,y,z) = x+y+z$ and $\sigma : t \mapsto (\sin(t), \cos(t), t)$, $t \in [0, 2\pi]$.

- We have

$$\frac{d\sigma}{dt} = (\cos(t), -\sin(t), 1) \quad .$$

Thus,

$$\left\| \frac{d\sigma}{dt} \right\| = \sqrt{\cos^2(t) + \sin^2(t) + 1} = \sqrt{2}$$

and so

$$\begin{aligned} \int_\sigma f\,ds &= \int_0^{2\pi} f(\sigma(t)) \left\| \frac{d\sigma}{dt}(t) \right\| dt \\ &= \int_0^{2\pi} (\cos(t) + \sin(t) + t)\sqrt{2}dt \\ &= \sqrt{2}\left(\sin(t) - \cos(t) + \frac{1}{2}t^2 \right)\Big|_0^{2\pi} \\ &= 2\sqrt{2}\pi^2 \end{aligned}$$

□

7.1.2. Evaluate the path integral

$$\int_{\mathcal{C}} f\,ds$$

where $f(x,y,z) = yz$ and $\mathcal{C}$ is the curve parameterized by $\sigma : t \mapsto (t, 3t, 2t)$, $t \in [1,3]$.

- Now

$$\frac{d\sigma}{dt} = (1, 3, 2)$$

and so

$$\left\| \frac{d\sigma}{dt} \right\| = \sqrt{1+9+4} = \sqrt{14}$$

We have

$$\begin{aligned} \int & \\ &= \int_1^3 (3t)(2t)\sqrt{14}dt \\ &= \int_1^3 6\sqrt{14}t^2\,dt \\ &= 2\sqrt{14}t^3\Big|_1^3 \\ &= 52\sqrt{14} \end{aligned}$$

□

Section 7.2

7.2.1. Let $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Evaluate the line integral of $\mathbf{F}$ along the path $\sigma(t) = (t,t,t)$, $0 \le t \le 1$.

- We have

$$\begin{aligned}
\int_\sigma \mathbf{F}\cdot d\mathbf{s} &= \int_0^1 \mathbf{F}(t,t,t)\cdot \frac{d\sigma}{dt}\,dt \\
&= \int_0^1 (t,t,t)\cdot(1,1,1)\,dt \\
&= \int_0^1 3t\,dt \\
&= \frac{3}{2}
\end{aligned}$$

□

7.2.2. Consider the force $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Compute the work done in moving along the parabola $y = x^2$, $z = 0$, from $x = -1$ to $x = 2$.

- We can parameterize the parabola via the path

$$\sigma(t) = \left(t, t^2, 0\right) \quad , \quad t \in [-1, 2]$$

We then have

$$\begin{aligned}
\int_\sigma \mathbf{F}\cdot d\mathbf{s} &= \int_{-1}^2 \mathbf{F}\left(t,t^2,0\right)\cdot \frac{d\sigma}{dt}(t)dt \\
&= \int_{-1}^2 \left(t,t^2,0\right)\cdot(1,2t,0)\,dt \\
&= \int_{-1}^2 \left(t+2t^3\right)dt \\
&= \left(\frac{1}{2}t^2 + \frac{1}{2}t^4\right)\Big|_{-1}^2 \\
&= 2 + 8 - \frac{1}{2} - \frac{1}{2} \\
&= 9
\end{aligned}$$

□

Section 7.3

7.3.1. Find the equation of the tangent plane to the parameterized surface $\Phi(u,v) = \left(2u, u^2+v, v^2\right)$ at the point (0,1,1).

- The values of u and v corresponding to the point (0,1,1) are $u=0$ and $v=1$. The normal vector to the tangent plane at (0,1,1) is thus given by

$$\begin{aligned} \mathbf{n}(0,1) &= \left.\frac{\partial\Phi}{\partial u}\right|_{(0,1)} \times \left.\frac{\partial\Phi}{\partial v}\right|_{(0,1)} \\ &= (2,2u,0)\left.\right|_{(0,1)} \times (0,1,2v)\left.\right|_{(0,1)} \\ &= (2,0,0)\times(0,1,2) \\ &= (0-0, 0-4, 2-0) \\ &= (0,-4,2) \end{aligned}$$

The equation of the plane through the point $(0,1,1)$ normal to the direction $\mathbf{n} = (0,-4,2)$ is

$$0 = \mathbf{n}\cdot((x,y,z)-(0,1,1) = -4(y-1)+2(z-1) = 2\,(-2y+1+z)$$

or

$$z = 2y-1$$

□

7.3.2. Find an expression for the unit vector normal to the parameterized surface

$$\Phi(u,v) = (\cos(v)\sin(u), \sin(v)\sin(u), \cos(u)) \qquad , \qquad (u,v)\in[0,\pi]\times[0,2\pi] \qquad .$$

Identify this surface.

- A normal vector to surface at the point $\Phi(u,v)$ is given by

$$\begin{aligned} \mathbf{n}(u,v) &= \left.\frac{\partial\Phi}{\partial u}\right|_{(u,v)} \times \left.\frac{\partial\Phi}{\partial v}\right|_{(u,v)} \\ &= (\cos(v)\cos(u), \sin(v)\cos(u), -\sin(u)) \times (-\sin(v)\sin(u), \cos(v)\sin(u), 0) \\ &= \left(0+\cos(v)\sin^2(u), \sin(v)\sin^2(u) - 0, \cos(u)\sin(u)\cos^2 v - \cos(u)\sin(u)\sin^2(v)\right) \\ &= \left(\cos(v)\sin^2(u), \sin(v)\sin^2(u), \cos(u)\sin(u)\right) \\ &= \sin(u)\,(\cos(v)\sin(u), \sin(v)\sin(u), \cos(u)) \end{aligned}$$

We have

$$\begin{aligned} \|\mathbf{n}\|^2 &= \sin^2(u)\left(\cos^2(v)\sin^2(v) + \sin^2(u)\sin^2(v) + \cos^2(u)\right) \\ &= \sin^2(u)\left(\sin^2(u)+\cos^2(u)\right) \\ &= \sin^2(u) \end{aligned}$$

Therefore, the *unit normal vector* at $\Phi(u,v)$ is

$$\frac{\mathbf{n}(u,v)}{\|\mathbf{n}(u,v)\|} = (\cos(v)\sin(u), \sin(v)\sin(u), \cos(u))$$

The surface S is that of the unit sphere. □

Section 7.4

7.4.1. Find the surface area of the unit sphere S represented parametrically by

$$\Phi(\theta,\phi) = (\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)) \qquad , \qquad (\theta,\phi) \in [0,2\pi]\times[0,\pi] \qquad .$$

- In the problem 7.3.5. we calculated the normal vector $\mathbf{n}$ to the surface prescribed by Φ. It remains to calculate the integral of $\|\mathbf{n}\|$ over the domain of Φ

$$\begin{aligned} A(S) &= \int_D \|\mathbf{n}\| dS \\ &= \int_0^\pi \int_0^{2\pi} \sin(\phi) d\theta\, d\phi \\ &= \int_0^\pi 2\pi \sin(\phi) \\ &= -2\pi\cos(\phi)\, \Big|_0^\pi \\ &= 2\pi\left(-(-1)+(1)\right) \\ &= 4\pi \end{aligned}$$

□

7.4.2. Let $\Phi(u,v) = (u-v, u+v, uv)$ and let D be the unit disk in the uv plane. Find the area of $\Phi(D)$.

- We have

$$\begin{aligned} \mathbf{n}(u,v) &= \frac{\partial\Phi}{\partial u}\times\frac{\partial\Phi}{\partial v} \\ &= (1,1,v)\times(-1,1,u) \\ &= (u-v, -v-u, 2) \end{aligned}$$

Thus,

$$\begin{aligned} \|\mathbf{n}\| &= \sqrt{(u-v)^2 + (-u-v)^2 + 4} \\ &= \sqrt{2\left(u^2+v^2+2\right)} \end{aligned}$$

and

$$\begin{aligned} A(S) &= \int_D \|\mathbf{n}\| dA \\ &= \int_D \sqrt{2\left(u^2+v^2+2\right)} dA \end{aligned}$$

To carry out this integral over the unit disk, we make a change of variables to polar coordinates

$$u = r\cos(\theta)$$
$$v = r\sin(\theta)$$

Recall that the Jacobian of the this transformation is r. Thus,

$$\begin{aligned}
A(S) &= \int_D \sqrt{2\left(u^2+v^2+2\right)}dA \\
&= \int_0^1 \int_0^{2\pi} \sqrt{2\left(r^2+2\right)}\, r\, d\theta\, dr \\
&= \int_0^1 2\pi \sqrt{2\left(r^2+2\right)} r\, dr \\
&= 2\pi \int_4^6 \sqrt{\tau}\frac{d\tau}{4} \qquad , \qquad \text{where} \tau = 2r^2+4 \\
&= \frac{\pi}{2}\frac{2}{3}\tau^{3/2}\Big|_4^6 \\
&= \frac{\pi}{3}\left(6\sqrt{6}-8\right)
\end{aligned}$$

□

Section 7.5

7.5.1. Evaluate $\int_S z\,dS$ where S is the upper hemisphere of radius a, that is, the set

$$\left\{(x,y,z) \in \mathbb{R}^3 \mid z = \sqrt{a^2+x^2+y^2}\right\} \qquad .$$

- We parameterize the upper hemisphere S via the map

$$\Phi(\theta,\phi) = (a\cos(\theta)\sin(\phi), a\sin(\theta)\sin(\phi), a\cos(\phi))$$

with $0 \le \theta \le 2\pi$, $0 \le \phi \le \frac{\pi}{2}$. We have already calculated the normal vector $\mathbf{n}$ associated with the analogous parameterization of the unit sphere. (See problem 7.3.5.) Reviewing that calculation, it becomes obvious that for a (hemi-) sphere of radius a we would have

$$\|\mathbf{n}\| = a^2\sin(\phi) \quad .$$

Thus,

$$\begin{aligned}
\int_S z\,dS &= \int_0^{\frac{\pi}{2}}\int_0^{2\pi} a\cos(\phi)\|\mathbf{n}\|d\theta\,d\phi \\
&= \int_0^{\frac{\pi}{2}}\int_0^{2\pi} a^3\cos(\phi)\sin(\phi)d\theta\,d\phi \\
&= 2\pi\int_0^{\frac{\pi}{2}} a^3\sin(\phi)\cos(\phi)\,d\phi \\
&= 2\pi a^3\int_0^1 u\,du \\
&= \pi a^3
\end{aligned}$$

□

Section 7.6

7.6.1. Let the temperature of a point in $\mathbb{R}^3$ be given by $3x^2+3z^2$. Compute the heat flux across the surface $x^2+z^2=2$, $0 \leq y \leq 2$ if $k=1$

- According to the text, the heat will flow with the vector field
$$\mathbf{F} = -k\nabla T$$
and so the heat flux across the surface S defined above, with $k=1$, will be
$$\int_S \mathbf{F} \cdot d\mathbf{S} = -\int_S \nabla T \cdot d\mathbf{S}$$
To calculate the integral on the right hand side we need to first find a suitable parameterization of the surface S. It is pretty obvious that S is just the surface of a cylinder of radius $\sqrt{2}$ with axis of symmetry coinciding along the y axis and with its top and bottom removed. We therefore use cylindrical coordinates to parameterize S:
$$\Phi(\theta, y) = \left(\sqrt{2}\cos(\theta), y, \sqrt{2}\sin(\theta)\right) \quad (\theta, y) \in [0, 2\pi] \times [0, 2] \quad .$$
We then have
$$\begin{aligned} \mathbf{n}(\theta, y) &= \left.\frac{\partial \Phi}{\partial \theta}\right|_{(\theta,y)} \times \left.\frac{\partial \Phi}{\partial y}\right|_{(\theta,y)} \\ &= \left(-\sqrt{2}\sin(\theta), 0, \sqrt{2}\cos(\theta)\right) \times (0,1,0) \\ &= \left(-\sqrt{2}\cos(\theta), 0, -\sqrt{2}\sin(\theta)\right) \end{aligned}$$
Hence,
$$\begin{aligned} \int_S \mathbf{F} \cdot d\mathbf{S} &= \int_S -\nabla T \cdot d\mathbf{S} \\ &= \int_S (-6x, 0, -6z) \cdot d\mathbf{S} \\ &= \int_0^2 \int_0^{2\pi} \left(-6\sqrt{2}\cos(\theta), 0, -6\sqrt{2}\sin(\theta)\right) \cdot \left(-\sqrt{2}\cos(\theta), 0, -\sqrt{2}\sin(\theta)\right) d\theta\, dy \\ &= \int_0^2 \int_0^{2\pi} \left(12\cos^2(\theta) + 12\sin^2(\theta)\right) d\theta\, dy \\ &= \int_0^2 \int_0^{2\pi} 12\, d\theta\, dy \\ &= \int_0^2 24\pi\, dy \\ &= 48\pi \end{aligned}$$

□