Math 4013
Solutions to Homework Problems from Chapter 6

Section 6.1

6.1.1. Let $S^* = (0,1] \times [0, 2\pi)$ and defined $T(r,\theta) = (r\cos(\theta), r\sin(\theta))$. Determine the image set S and show that T is one-to-one on S^*.

- To find the image of S^* under T, we first calculate the image of the boundary of S^*.
  Now the boundary of S^* consists of the the following 4 curves

$$\begin{array}{rcl} \sigma_1(t) & : & t \mapsto (t,0) \quad , \quad t \in [0,1] \\ \sigma_2(t) & : & t \mapsto (1,t) \quad , \quad t \in [0,2\pi] \\ \sigma_3(t) & : & t \mapsto (1-t, 2\pi) \quad , \quad t \in [0,1] \\ \sigma_4(t) & : & t \mapsto (0, 2\pi - t) \quad , \quad t \in [0,2\pi] \end{array}$$

  Note the curves σ_3 and σ_4, while part of the boundary of S^* do not belong to S^*.
  The images of these four curves under the map T are given by

$$\begin{array}{rcl} \gamma_1(t) & : & t \mapsto T(\sigma_1(t)) = (t,0) \quad , \quad t \in [0,1] \\ \gamma_2(t) & : & t \mapsto T(\sigma_2(t)) = (\cos(t), \sin(t)) \quad , \quad t \in [0,2\pi] \\ \gamma_3(t) & : & t \mapsto T(\sigma_3(t)) = (1-t, 0) \quad , \quad t \in [0,1] \\ \gamma_4(t) & : & t \mapsto T(\sigma_4(t)) = (0,0) \quad , \quad t \in [0,2\pi] \end{array}$$

  Thus, the image of γ_1 is portion of the x-axis between 0 and 1, the image of γ_1 is the unit circle, the image of γ_3 is the portion of the x-axis between 0 and 1, and the image of γ_4 is the origin.
  It would appear that the portion of the x-axis between 0 and 1 is counted twice - however recall that the curve σ_3 does not lie in the domain of T. Nor does the curve σ_4. Therefore, the image curves γ_3 and γ_4 are not to be considered as being part of S. We conclude that the image of S^* by T is the unit disk minus the origin.
  To show that the map T is one-to-one, we must show ı(i) that T is surjective; i.e., every point of the unit disc is the image of the form $(x,y) = T(r,\theta)$ for some $(r,\theta) \in S^*$.
  This is already evident from the definition of S;

$$S := \left\{ (x,y) \in \mathbb{R}^2 \mid (x,y) = T(r,\theta) \quad , \quad \text{for some}(r,\theta) \in S^* \right\} \quad .$$

  ı(ii) that T is injective; i.e., if $T(r,\theta) = T(r',\theta')$ then $(r,\theta) = (r',\theta')$.
  Well, suppose

$$(r\cos(\theta), r\sin(\theta)) = (r'\cos(\theta'), r'\sin(\theta'))$$

  This can happen if and only if

$$r = r' = 0$$

  or

$$r = r' \quad \text{and} \quad \theta = \theta' + 2\pi n \quad , \quad n \in \mathbb{Z} \quad .$$

  But $r = r' = 0$ is excluded from S^*, and there are no two (r,θ), (r',θ') in S^* for which $\theta = \theta' + 2n\pi$.
  Hence, the transformation T is one-to-one. □

6.1.2. Let $D^* = [0,1] \times [0,1]$ and define T on D^* by $T(u,v) = \left(-u^2 + 4u, v\right)$. Find D. Is T one-to-one?

- The region D^* is the region in the uv-plane bounded by the lines

$$\begin{array}{rcl} \sigma_1(t) & = & (t,0) \quad , \quad t \in [0,1] \\ \sigma_2(t) & = & (1,t) \quad , \quad t \in [0,1] \\ \sigma_3(t) & = & (t,1) \quad , \quad t \in [0,1] \\ \sigma_4(t) & = & (0,t) \quad , \quad t \in [0,1] \end{array}$$

The region $D = T(D)$ should therefore be the region in the xy-plane bounded by the curves

$$\begin{aligned}
\gamma_1(t) &= T(\sigma_1(t)) = \left(-t^2+4t, 0\right) \quad , \quad t \in [0,1] \\
\gamma_2(t) &= T(\sigma_2(t)) = (3,t) \quad , \quad t \in [0,1] \\
\gamma_3(t) &= T(\sigma_3(t)) = \left(-t^2+4t, 1\right) \quad , \quad t \in [0,1] \\
\gamma_4(t) &= T(\sigma_4(t)) = (0,t) \quad , \quad t \in [0,1]
\end{aligned}$$

The curve γ_1 is the line segment along the x-axis between $(0,0)$ and $(3,0)$, γ_2 corresponds to the vertical line segment between the points $(3,0)$ and $(3,1)$, γ_3 corresponds to the horizontal line segment between the points $(3,1)$ and $(0,1)$, and γ_4 corresponds to the vertical line segment between $(0,0)$ and $(0,1)$.

Thus, $S = [0,3] \times [0,1]$.

By definition $T : S^* \to S$ is surjective. We check to see that T is injective. If

$$\left(-u^2+4u, v\right) = \left(-u'^2+4u', v'\right)$$

then we must have

$$\begin{aligned}
u' &= \tfrac{4 \pm \sqrt{16+4(u^2-4u)}}{2} = 2 \pm (u-2) = \begin{cases} u \\ 4-u \end{cases} \\
v' &= v
\end{aligned}$$

But if $u \in [0,1]$, $u' = 4 - u \neq [0,1]$, So it is not possible for two distinct points in S^* to land on the same point in S under the map T. Thus, T is both surjective and injective; hence T is one-to-one. $\square$

6.1.3. Let $D^* = [0,1] \times [0,1]$ and define T on D^* by $T(u,v) = (uv, u)$. Find D. Is T one-to-one? If not, can we eliminate some subset of D^* so that on the remainder T is one-to-one?

- The region D^* is the region in the uv-plane bounded by the lines

$$\begin{aligned}
\sigma_1(t) &= (t,0) \\
\sigma_2(t) &= (1,t) \\
\sigma_3(t) &= (t,1) \\
\sigma_4(t) &= (0,t)
\end{aligned}$$

The region $D = T(D)$ should therefore be the region in the xy-plane bounded by the curves

$$\begin{aligned}
\gamma_1(t) &= T(\sigma_1(t)) = (0,t) \\
\gamma_2(t) &= T(\sigma_2(t)) = (t,1) \\
\gamma_3(t) &= T(\sigma_3(t)) = (t,t) \\
\gamma_4(t) &= T(\sigma_4(t)) = (0,0)
\end{aligned}$$

Thus, the image of the curve σ_1 coincides with the y-axis; the image of the curve σ_2 is just the horizontal line $y = 1$; the image of the curve σ_3 coincides with the diagonal line $y = x$; and image of the curve σ_4 is just the point (0,0). Thus, D coincides with the interior of the triangle with vertices (0,0), (0,1), and (1,1).

This map cannot be one-to-one since every point along the curve σ_4 is mapped to the point (0,0). However, if we remove this curve from the domain of T, then the map becomes one-to-one. $\square$

6.1.4. Let $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is a 2×2 matrix. Show that T is one-to-one if and only if the determinant of $\mathbf{A}$ is non-zero.

- If T is one-to-one then it must have an inverse. However, a matrix $\mathbf{A}$ has an inverse if and only if its determinant is non-zero. Thus, T is one-to-one if and only if $det\ \mathbf{A} \neq 0$. $\square$

6.1.5. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear; i.e, $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a 2×2 matrix. Show that if $det\ \mathbf{A} \neq 0$, then T takes parallelograms to parallelograms. (Hint: any parallelogram in $\mathbb{R}^2$ can be described as a set $\{\mathbf{r} = \mathbf{p} + \lambda\mathbf{v} + \mu\mathbf{w} \mid \lambda, \mu \in [0,1]\}$ where $\mathbf{p}$, $\mathbf{v}$, $\mathbf{w}$ are suitable vectors in $\mathbb{R}^2$ with $\mathbf{v}$ not a scalar multiple of $\mathbf{w}$.

- Suppose

$$P = \left\{ \mathbf{r} \in \mathbb{R}^2 \mid \mathbf{r} = \mathbf{p} + \lambda \mathbf{v} + \mu \mathbf{w} \mid \lambda, \mu \in [0,1] \right\}$$

is a parallelogram. Then $T(P)$ is

$$\begin{aligned} T(P) &= \left\{ \mathbf{r}' \in \mathbb{R}^2 \mid \mathbf{r}' = \mathbf{A}(\mathbf{p} + \lambda \mathbf{v} + \mu \mathbf{w}) \quad , \quad \lambda, \mu \in [0,1] \right\} \\ &= \left\{ \mathbf{r}' \in \mathbb{R}^2 \mid \mathbf{r}' = \mathbf{A}\mathbf{p} + \mathbf{A}(\lambda \mathbf{v}) + \mathbf{A}(\mu \mathbf{w}) \quad , \quad \lambda, \mu \in [0,1] \right\} \\ &= \left\{ \mathbf{r}' \in \mathbb{R}^2 \mid \mathbf{r}' = (\mathbf{A}\mathbf{p}) + \lambda(\mathbf{A}\mathbf{v}) + \mu(\mathbf{A}\mathbf{w}) \quad , \quad \lambda, \mu \in [0,1] \right\} \\ &= \left\{ \mathbf{r}' \in \mathbb{R}^2 \mid \mathbf{r}' = \mathbf{p}' + \lambda \mathbf{v}' + \mu \mathbf{w}') \quad , \quad \lambda, \mu \in [0,1] \right\} \end{aligned}$$

where $\mathbf{p}' = \mathbf{A}\mathbf{p}$, $\mathbf{v}' = \mathbf{A}\mathbf{v}$, and $\mathbf{w}' = \mathbf{A}\mathbf{w}$. If we can demonstate that $\mathbf{v}' \neq t\mathbf{w}'$, for any $t \in \mathbb{R}$, then we may conclude that $T(P)$ is a parallelogram.

We argue as follows. Suppose $\mathbf{v}' = t\mathbf{w}'$. Then

$$0 = \mathbf{v}' - t\mathbf{w} = \mathbf{A}\mathbf{v} - t\mathbf{A}\mathbf{w} = \mathbf{A}(\mathbf{v} - t\mathbf{w}) \quad .$$

Since $\mathbf{v}$ and $\mathbf{w}$ are not scalar multiples of one another, $\mathbf{v} - \mathbf{tw}$ must be a non-zero vector. But if the matrix $\mathbf{A}$ maps a non-zero vector to zero, it must be singular; hence, det $\mathbf{A} = 0$. But hypothesis, *det* $\mathbf{A} \neq 0$. Therefore, $\mathbf{v}' \neq t\mathbf{w}'$. Hence, $T(P)$ is a parallelogram. $\square$

Section 6.2

6.2.1. Let D be the unit circle. Evaluate

$$\int_D \exp\left(x^2+y^2\right) dx\, dy$$

by making a change of variables to polar coordinates.

- The coordinate transformation

$$T : (r,\theta) \mapsto (r\cos(\theta), r\sin(\theta))$$

maps the rectangle $R = \{0 \le r \le 1\ ,\ 0 \le \theta < 2\pi\}$ onto the unit circle. The Jacobian of this transformation is

$$\begin{aligned} J(T) &= \left|\det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{pmatrix}\right| \\ &= \left|\frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial y}{\partial r}\frac{\partial x}{\partial \theta}\right| \\ &= \left|(\cos(\theta))\,(r\cos(\theta)) - (\sin(\theta))\,(-r\sin(\theta))\right| \\ &= \left|r\left(\cos^2(\theta)+\sin^2(\theta)\right)\right| \\ &= r \end{aligned}$$

Thus, by the change of variables formula

$$\begin{aligned} \int_D \exp\left(x^2+y^2\right) dx\, dy &= \int_R \exp\left(r^2\right) J(T) dr\, d\theta \\ &= \int_0^1 \int_0^{2\pi} e^{r^2} r\, d\theta\, dr \\ &= 2\pi \int_0^1 re^{r^2}\, dr \\ &= 2\pi\left(\frac{1}{2}\int_0^1 e^u\, du\right) \\ &= \pi\,(e-1) \end{aligned}$$

□

6.2.2. Let D be the region $0 \le y \le x$ and $0 \le x \le 1$. Evaluate

$$\int_D (x+y) dx\, dy$$

by making the change of variables $x = u+v$, $y = u-v$. Check your answer by evaluating the integral directly by using an iterated integral.

- Let T be the coordinate transformation defined by

$$T : (u,v) \mapsto (u+v, u-v)$$

The Jacobian of this transformation is

$$\begin{aligned} J(T) &= \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right| \\ &= \left|(1)(-1)-(1)(1)\right| \\ &= 2 \end{aligned}$$

To find the pre-image D^* of the region D by T, we first calculate the inverse of T. Solving

$$\begin{aligned} x &= u+v \\ y &= u-v \end{aligned}$$

for u and v yields

$$\begin{aligned} u &= \frac{1}{2}(x+y) \\ v &= \frac{1}{2}(x-y) \quad . \end{aligned}$$

Thus, the pre-images of the three boundary curves of D

$$\begin{aligned} \sigma_1(t) &: t \mapsto (t,0) \quad , \quad t \in [0,1] \\ \sigma_2(t) &: t \mapsto (1,t) \quad , \quad t \in [0,1] \\ \sigma_2(t) &: t \mapsto (t,t) \quad , \quad t \in [0,1] \end{aligned}$$

will be

$$\begin{aligned} \gamma_1(t) &= T^{-1}(\sigma_1) = \left(\frac{1}{2}t, \frac{1}{2}t\right) \quad , \quad t \in [0,1] \\ \gamma_2(t) &= T^{-1}(\sigma_2) = \left(\frac{1}{2}(1+t), \frac{1}{2}(1-t)\right) \quad , \quad t \in [0,1] \\ \gamma_1(t) &= T^{-1}(\sigma_1) = (t,0) \quad , \quad t \in [0,1] \end{aligned}$$

The area in the uv-plane bounded by these three lines will be the triangle with vertices $(0,0)$, $\left(\frac{1}{2}, \frac{1}{2}\right)$, and $(1,0)$.

This region can be regarded as a region of type II.

$$D^* = \left\{0 \leq v \leq \frac{1}{2} \ , \ v \leq u \leq 1-v \quad .\right\}$$

Thus,

$$\begin{aligned} \int_D (x+y)dx\,dy &= \int_{D^*} 2uJ(T)du\,dv \\ &= \int_0^{\frac{1}{2}} \int_v^{1-v} 4u\,du\,dv \\ &= \int_0^{\frac{1}{2}} 2\left((1-v)^2 - v^2\right)dv \\ &= \int_0^{\frac{1}{2}} 2(1-2v)\,dv \\ &= 1 - \frac{1}{2} - 0 - 0 \\ &= \frac{1}{2} \end{aligned}$$

To check our result we shall integrate over D directly. Now

$$D = \{0 \leq x \leq 1 \ , \ 0 \leq y \leq x\}$$

so

$$\begin{aligned} \int_D (x+y)dx\,dy &= \int_0^1 \int_0^x (x+y)\,dy\,dx \\ &= \int_0^1 \left(xy + \frac{1}{2}y^2\right)\Big|_0^x dx \\ &= \int_0^1 \left(\frac{3}{2}x^2\right) dx \\ &= \frac{1}{2} \end{aligned}$$

$\square$

6.2.3. Let $T(u,v) = (x(u,v), y(u,v))$ be the mapping defined by $T(u,v) = (4u, 2u+3v)$. Let D^* be the region in $u-v$ plane corresponding to the rectangle $[0,1]\times[1,2]$. Find $D = T(D^*)$ and evaluate

(a) $\int_D xy\, dA$

(b) $\int_D (x-y) dA$

- To find the image D of the the region D^* in the xy-plane, we note the map $T(u,v)$ is linear in u and v. In Problem 6.2.10 we showed the image of a parallelogram in $\mathbb{R}^2$ (in particular, the image of a rectangle) by a linear map $T:\mathbb{R}^2 \to \mathbb{R}^2$ is always a parallelogram. To prescribe the image D of D^* by the map T is is therefore sufficient to present its four vertices. But these vertices will just be the images of the corners of D^* by T; thus, the region D will be the parallelogram in the xy-plane with vertices

$$\begin{aligned}
\mathbf{v}_1 &= T(0,1) = (0,3)\\
\mathbf{v}_2 &= T(1,1) = (4,5)\\
\mathbf{v}_3 &= T(1,2) = (4,8)\\
\mathbf{v}_4 &= T(0,2) = (0,6)
\end{aligned}$$

Let us now compute the Jacobian of the transformation.

$$\begin{aligned}
J(T) &= \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right|\\
&= |(4)(3)-(2)(0)|\\
&= 12
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_D xy\, dx\, dy &= \int_{D^*} (4u)(2u+3v)J(T)du\, dv\\
&= 24\int_0^1\int_1^2 \left(4u^2+6uv\right) dv\, du\\
&= 24\int_0^1 \left(4u^2v+3uv^2\right)\Big|_1^2 du\\
&= 24\int_0^1 \left(8u^2+12u-4u^2-3u\right) du\\
&= 24\left(\frac{4}{3}+\frac{9}{2}\right)\\
&= 32+108\\
&= 140
\end{aligned}$$

and

$$\begin{aligned}
\int_D (x-y)\,dx\,dy &= \int_{D^*} (4u-2u-3v)J(T)du\,dv \\
&= 12\int_0^1\int_1^2 (2u-3v)\,dv\,du \\
&= 12\int_0^1 \left(2uv-\frac{3}{2}v^2\right)\Big|_1^2 du \\
&= 12\int_0^1 \left(4u-6-2u+\frac{3}{2}\right) du \\
&= 12\left(u^2-\frac{9}{2}\right)\Big|_0^1 \\
&= 12-54 \\
&= -42
\end{aligned}$$

$\square$

6.2.4. Define $T(u,v)=\left(u^2-v^2,2uv\right)$. Let D^* be the set of (u,v) with $u^2+v^2\leq 1$, $u\geq 0$, $v\geq 0$. Find $T(D^*)=D$. Evaluate

$$\int_D dA \quad .$$

- To find the image of D^*, we calculate the images of the boundary curves of D^*. The following three curves form the boundary of D^*

$$\begin{aligned}
\sigma_1(t) &= (t,0) \quad , \quad t\in[0,1] \\
\sigma_2(t) &= (\cos(t),\sin(t)) \quad , \quad t\in\left[0,\frac{\pi}{2}\right] \\
\sigma_3(t) &= (0,t) \quad , \quad t\in[0,1]
\end{aligned}$$

The images of these curves under the map T are given by

$$\begin{aligned}
\gamma_1(t) &= T(\sigma_1(t))=\left(t^2,0\right) \quad , \quad t\in[0,1] \\
\gamma_2(t) &= T(\sigma_2(t))=\left(\cos^2(t)-\sin^2(t),2\cos(t)\sin(t)\right) \quad , \quad t\in\left[0,\frac{\pi}{2}\right] \\
\gamma_3(t) &= T(\sigma_3(t))=\left(-t^2,0\right) \quad , \quad t\in[0,1]
\end{aligned}$$

These curves bound the region D pictured below:

The Jacobian of the transformation T is

$$\begin{aligned}
J(t) &= \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right| \\
&= |(2u)(2u)-(-2v)(2v)| \\
&= 4\left(u^2+v^2\right)
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_D dy\,dy &= \int_{D^*} J(T)du\,dv \\
&= \int_{D^*} 4\left(u^2+v^2\right)du\,dv
\end{aligned}$$

Since D^* is the unit disc, this last integral will be evaluated most easily if we make another change of variables to polar coordinates:

$$\begin{aligned}
u &= r\cos(\theta) \\
v &= r\sin(\theta)
\end{aligned}$$

The Jacobian of this transformation is r. Thus,

$$\begin{aligned}\int_{D^*} 4\left(u^2+v^2\right) du\, dv &= \int_0^1 \int_0^{2\pi} \left(4r^2\right) r\, d\theta\, dr \\ &= 8\pi \int_0^1 r^3 dr \\ &= 2\pi\end{aligned}$$

□

6.2.5. Let $T(u,v)$ be as in Exercise 6.2.4. By making this change of variables, evaluate

$$\int_D \frac{dA}{\sqrt{x^2+y^2}}$$

- The coordinate transformation in 6.2.4. is given by

$$\begin{aligned} x &= u^2 - v^2 \\ y &= 2uv \end{aligned}$$

and so the associated Jacobian is

$$\begin{aligned} J(T) &= \left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix}\right| \\ &= \left|\det\begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix}\right| \\ &= 2u^2 + 2v^2 \end{aligned}$$

$$\begin{aligned}\int_D \frac{dA}{\sqrt{x^2+y^2}} &= \int_{D^*} \frac{1}{\sqrt{(u^2-v^2)+(2uv)^2}} J(T) \\ &= \int_{-1}^1 \int_0^{\sqrt{1-u^2}} \frac{2u^2+2v^2}{\sqrt{u^4+2u^2v^2+v^4}} dv du \\ &= \int_{-1}^1 \int_0^{\sqrt{1-u^2}} \frac{2u^2+2v^2}{\sqrt{(u^2+v^2)^2}} dv du \\ &= \int_{-1}^1 \int_0^{\sqrt{1-u^2}} 2 dv du \\ &= \int_{-1}^1 2\sqrt{1-u^2} du \\ &= \left(x\sqrt{(1-x^2)} + \arcsin x\right)\Big|_{-1}^1 \\ &= 0 + \frac{\pi}{2} - \left(0 + \left(-\frac{\pi}{2}\right)\right) \\ &= \pi \end{aligned}$$

□

6.2.6. Integrate $ze^{x^2+y^2}$ over the cylinder $x^2+y^2 \le 4$, $-2 \le z \le 3$.

- The cylider $D = \left\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \le 4\ ,\ -2 \le z \le 3\right\}$ is the image of the rectangle $D^* = \left\{(r,\theta,z) \in \mathbb{R}^3 \mid 0 \le r \le 2\ ,\ 0 \le \theta < 2\pi\ ,\ -2 \le z \le 3\right\}$ under the (polar) coordinate transformation

$$T : (r,\theta,z) \ \mapsto (r\cos(\theta), r\sin(\theta), z) \quad .$$

The Jacobian of this transformation is

$$\begin{aligned}
J(T) &= \left|\det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{pmatrix}\right| \\
&= \left|\det\begin{pmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}\right| \\
&= \left|r\left(\cos^2(\theta) + \sin^2(\theta)\right)\right| \\
&= r
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_D z e^{x^2+y^2} dA &= \int_{D^*} z e^{r^2} J(T) dA \\
&= \int_{-2}^{3}\int_0^2\int_0^{2\pi} z r e^{r^2}\, d\theta\, dr\, dz \\
&= 2\pi \int_{-2}^{3}\int_0^2 z r e^{r^2}\, dr\, dz \\
&= \pi \int_{-2}^{3}\int_0^4 z e^u\, du\, dz \\
&= \pi \int_{-2}^{3} z\left(e^4 - 1\right) dz \\
&= \frac{\pi}{2}\left(e^4 - 1\right)\left(3^2 - (-2)^2\right) \\
&= \frac{5\pi}{2}\left(e^4 - 1\right)
\end{aligned}$$

□