Math 4013
Solutions to Homework Problems from Chapter 5

Section 5.1

5.2.1. Evaluate the following interated integrals.

(a)

- $$\int_{-1}^{1}\int_{0}^{1}\left(x^4y+y^2\right)dy\,dx$$

$$\begin{aligned}
\int_{-1}^{1}\int_{0}^{1}\left(x^4y+y^2\right)dy\,dx &= \int_{-1}^{1}\left[\int_{0}^{1} dy\left(x^4y+y^2\right)dy\right]dx \\
&= \int_{-1}^{1}\left(\frac{1}{2}x^4y^2+\frac{1}{3}y^3\right)\Bigg|_0^1 dx \\
&= \int_{-1}^{1}\left(\frac{1}{2}x^4+\frac{1}{3}\right)dx \\
&= \left(\frac{1}{10}x^5+\frac{1}{3}x\right)\Bigg|_{-1}^{1} \\
&= \frac{1}{10}+\frac{1}{3}-\left(-\frac{1}{10}-\frac{1}{3}\right) \\
&= \frac{13}{15}
\end{aligned}$$

□

(b)

- $$\int_{0}^{\pi/2}\int_{0}^{1}\left(y\cos(x)+2\right)dy\,dx$$

$$\begin{aligned}
\int_{0}^{\pi/2}\int_{0}^{1}\left(y\cos(x)+2\right)dy\,dx &= \int_{0}^{\pi/2}\left[\int_{0}^{1}\left(y\cos(x)+2\right)dy\right]dx \\
&= \int_{0}^{\pi/2}\left(\frac{1}{2}y^2\cos(x)+2y\right)\Bigg|_0^1 dx \\
&= \int_{0}^{\pi/2}\left(\frac{1}{2}\cos(x)+2\right)dx \\
&= \left(\frac{1}{2}\sin(x)+2x\right)\Bigg|_0^{\pi/2} \\
&= \frac{1`}{2}+\pi-0-0 \\
&= \frac{1}{2}+\pi
\end{aligned}$$

□

5.2.1(a). Evaluate the integral in 5.2.1(a) by integrating first with respect to x and then with respect to y.

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\begin{aligned}
\int_0^1 \left[\int_{-1}^1 \left(x^4 y + y^2\right) dx\right] dy &= \int_0^1 \left(\frac{1}{5}x^4 y + y^2 x\right)\Big|_{-1}^1 dy \\
&= \int_0^1 \left(\frac{1}{5}y + y^2 - \left(-\frac{1}{5}y - y^2\right)\right) dy \\
&= \int_0^1 \left(\frac{2}{5}y + 2y^2\right) dy \\
&= \left(\frac{1}{5}y^2 + \frac{2}{3}y^2\right)\Big|_0^1 \\
&= \frac{1}{5} + \frac{2}{3} \\
&= \frac{13}{15}
\end{aligned}
$$

□

5.2.1(b). Evaluate the integral in 5.2.1(b) by integrating first with respect to x and then with respect to y.

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$$
\begin{aligned}
\int_0^1 \left[\int_0^{\pi/2} (y\cos(x) + 2)\, dx\right] dy &= \int_0^1 (y\sin(x) + 2x)\ \Big|_0^{\pi/2} dy \\
&= \int_0^1 (y + \pi)\, dy \\
&= \left(\frac{1}{2}y^2 + \pi y\right)\Big|_0^1 \\
&= \frac{1}{2} + \pi
\end{aligned}
$$

□

5.2.3. (a) Demonstrate informally that the volume of the solid of revolution shown in Figure 5.1.13. is

$$\pi \int_a^b [f(x)]^2\, dx \quad .$$

- To calculate the volume of a solid of revolution, we first imagine partitioning the interval $[a, b]$ into n subintervals of width

  $$\Delta x = \frac{b-a}{n} \quad .$$

  This will induce a corresponding partition of the solid of revolution; each slice of which looking pretty much like a cylinder of length Δx and radius $f(x)$. Since the volume of a cylinder is given by

  $$Vol_{cyl} = \pi r^2 \ell$$

  we see that the contribution of the i^{th} slice to the total volume of the cylinder will be

  $$\Delta V_i = \pi \left(f(x_i)\right)^2 \Delta x$$

  where $x_i \in [a + (n-1)\Delta x, a + n\Delta x]$ (that is to say, x_i is point in the n^{th} subinterval of $[a, b]$). The total volume of the solid of revolution is thus approximated by the Riemann sum

  $$Vol \approx \sum_{i=1}^n \Delta V_i = \sum_{i=1}^n \pi \left(f(x_i)\right)^2 \Delta x$$

Taking the limit as n goes to infinity we can replace the Riemann sum by the corresponding Riemann integral, to obtain

$$Vol = \int_a^b \pi \left(f(x)\right)^2 dx \quad .$$

$\square$

(b) Show the volume of the region obtained by rotating the region under the graph of parabola $y = -x^2 + 2x + 3$, $-1 \leq x \leq 3$, about the x-axis is $512\pi/15$.

- Plugging into the formula "derived" in Part (a), we have

$$\begin{aligned}
Vol &= \int_{-1}^{3} \pi \left(-x^2 + 2x + 3\right)^2 dx \\
&= \int_{-1}^{3} \pi \left(x^4 - 4x^3 - 2x^2 + 12x + 9\right) dx \\
&= \pi \left(\frac{1}{5}x^5 - x^4 - \frac{2}{3}x^3 + 6x^2 + 9x\right)\Big|_{-1}^{3} \\
&= \frac{\pi}{15}\left(3x^5 - 15x^4 - 10x^3 + 90x^2 + 135x\right)\Big|_{-1}^{3} \\
&= \frac{\pi}{15}(729 - 1215 - 270 + 810 + 405 + 3 + 15 - 10 - 90 + 135) \\
&= \frac{512\pi}{15}
\end{aligned}$$

$\square$

5.1.4. Evaluate the following double integrals

(a)

$$\int_R \left(x^2y^2 + x\right) dxdy \quad , \quad R = [0,2] \times [-1,0]$$

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$$\begin{aligned}
\int_R \left(x^2y^2 + x\right) dxdy &= \int_0^2 \int_{-1}^0 \left(x^2y^2 + x\right) dy\, dx \\
&= \int_0^2 \left(\frac{1}{3}x^2y^3 + xy\right)\Big|_{-1}^{0} dx \\
&= \int_0^2 \left(0 + 0 - \left(-\frac{1}{3}x^2 - x\right)\right) dx \\
&= \int_0^2 \left(\frac{1}{3}x^2 + x\right) dx \\
&= \left(\frac{1}{9}x^3 + \frac{1}{2}x^2\right)\Big|_0^2 \\
&= \frac{8}{9} + \frac{4}{2} \\
&= \frac{26}{9}
\end{aligned}$$

$\square$

(b)

- $$\int_R \left(x^3 + y^3\right) dA \quad , \quad R = [0,1] \times [0,1]$$

$$\begin{aligned}
\int_R \left(x^3 + y^3\right) dA &= \int_0^1 \int_0^1 \left(x^3 + y^3\right) dy\, dx \\
&= \int_0^1 \left(x^3 y + \frac{1}{4} y^4\right)\bigg|_0^1 dx \\
&= \int_0^1 \left(x^3 + \frac{1}{4}\right) dx \\
&= \left(\frac{1}{4}x^4 + \frac{1}{4}x\right)\bigg|_0^1 \\
&= \frac{1}{4} + \frac{1}{4} + 0 + 0 \\
&= \frac{1}{2}
\end{aligned}$$

□

(c)

- $$\int_R y e^{xy}\; dA \quad , \quad R = [0,1] \times [0,1]$$

$$\begin{aligned}
\int_R y e^{xy}\; dA &= \int_0^1 \int_0^1 y e^{xy} dx dy \\
&= \int_0^1 \left(y\left(\frac{1}{y}e^{xy}\right)\right)\bigg|_0^1 dy \\
&= \int_0^1 \left(e^y - 1\right) dy \\
&= \left(e^y - y\right)\big|_0^1 \\
&= e - 1 - (1 - 0) \\
&= e - 2
\end{aligned}$$

□

(d)

- $$\int_R \left(x^m y^n\right) dA \quad , \quad R = [0,1] \times [0,1]$$

$$\begin{aligned}
\int_R \left(x^m y^n\right) dA &= \int_0^1 \int_0^1 \left(x^m y^n\right) dy\, dx \\
&= \int_0^1 \left(\frac{1}{n+1} x^m y^{n+1}\right)\bigg|_0^1 dx \\
&= \int_0^1 \frac{1}{n+1} x^m\, dx \\
&= \frac{1}{(n+1)(m+1)} x^{m+1}\bigg|_0^1 \\
&= \frac{1}{(n+1)(m+1)}
\end{aligned}$$

□

(e)

$$\int_R (ax+by+c)\,dA \quad , \quad R=[0,1]\times[0,1]$$

•

$$\begin{aligned}
\int_R (ax+by+c)\,dA &= \int_0^1\int_0^1 (ax+by+c)\,dy\,dx \\
&= \int_0^1 \left(axy+\frac{1}{2}by^2+cy\right)\Bigg|_0^1 dx \\
&= \int_0^1 \left(ax+\frac{b}{2}+c\right)dx \\
&= \left(\frac{1}{2}ax^2+\frac{b}{2}x+cx\right)\Bigg|_0^1 \\
&= \frac{a+b}{2}+c
\end{aligned}$$

□

5.2.5. Compute the volume of the solid bounded by the surface $z=\sin(y)$, the planes $x=1$, $x=0$, $y=0$, $y=\frac{\pi}{2}$, $z=0$.

- This solid is interpretable as the volume under the graph of $f(x,y)=\sin(y)$ and above the rectangle

$$R=\left\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 1 \quad , \quad 0\le y\le\frac{\pi}{2}\right\} \quad .$$

We can therefore apply the general formula

$$\begin{aligned}
Vol &= \int_R f(x,y)dA \\
&= \int_0^1 dx\int_0^{\pi/2} dy\,(\sin(y)) \\
&= \int_0^1 dx\left(-\cos\left(\frac{\pi}{2}\right)+\cos(0)\right) \\
&= \int_0^1 dx \\
&= 1
\end{aligned}$$

□

Section 5.3

5.3.1(a). Evaluate the following iterated integral and draw the region D determined by the limits of integration. State whether the region D is of type I, type II, or both.

$$\int_0^1 \int_0^{x^2} dy dx$$

- The region of integration is bounded by the curves

$$\begin{aligned} y &= x^2 \\ y &= 0 \\ x &= 1 \\ x &= 0 \end{aligned}$$

This is just the area under the parabola $y = x^2$ between $x = 0$ and $x = 1$.

Computing the iterated integral we get

$$\begin{aligned} \int_0^1 \int_0^{x^2} dy dx &= \int_0^1 y \Big|_0^{x^2} dx \\ &= \int_0^1 x^2 dx \\ &= \frac{1}{3} x^2 \Big|_0^1 \\ &= \frac{1}{3} \end{aligned}$$

□

5.3.1(b). Evaluate the following iterated integral and draw the region D determined by the limits of integration. State whether the region D is of type I, type II, or both.

$$\int_0^1 \int_1^{e^x} (x + y) dy\, dx$$

- The region of integration is bounded by the curves

$$\begin{aligned} y &= e^x \\ y &= 1 \\ x &= 1 \\ x &= 0 \end{aligned}$$

Computing the iterated integral we get

$$\begin{aligned} \int_0^1 \int_1^{e^x} (x + y) dy dx &= \int_0^1 \left(xy + \frac{1}{2} y^2 \right) \Big|_1^{e^x} dx \\ &= \int_0^1 \left(x e^x + \frac{1}{2} e^{2x} - x - \frac{1}{2} \right) dx \\ &= \left(x e^x - e^x + \frac{1}{4} e^{2x} - \frac{1}{2} x^2 - \frac{1}{2} x \right) \Big|_0^1 \\ &= e - e + \frac{e^2}{4} - \frac{1}{2} - \frac{1}{2} + 1 - \frac{1}{4} + 0 + 0 \\ &= \frac{e^2 - 1}{4} \end{aligned}$$

□

5.3.2. Use double integrals to compute the area of a circle of radius r.

- To find the area of a circle, we regard it as the region of type I bounded by the following four curves

$$\begin{aligned} y &= -\sqrt{r^2-x^2} \\ y &= \sqrt{r^2-x^2} \\ x &= -r \\ x &= r \end{aligned}$$

Thus,

$$\begin{aligned} A &= \int_{-r}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} dy\,dx \\ &= \int_{-r}^{r} y\,\Big|_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}}\,dx \\ &= \int_{-r}^{r} 2\sqrt{r^2-x^2}dx \\ &= 2\left(\frac{x}{2}\sqrt{r^2-x^2}+\frac{r^2}{2}\sin^{-1}\left(\frac{x}{r}\right)\right)\Big|_{-r}^{r} \\ &= 2\left(0-\frac{r^2}{2}\sin^{-1}(1)-0+\frac{r^2}{2}\sin^{-1}(-1)\right) \\ &= r^2\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right) \\ &= \pi r^2 \end{aligned}$$

□

5.3.3. Let D be the region bounded by the x and y axes and the line $3x+4y=10$. Compute

$$\int_D \left(x^2+y^2\right)dA \quad .$$

- To compute this integral we consider the region D as a region of type I bounded by the curves

$$\begin{aligned} y &= 0 \\ y &= \frac{10}{4}-\frac{3}{4}x \\ x &= 0 \\ x &= \frac{10}{3} \end{aligned}$$

Thus,

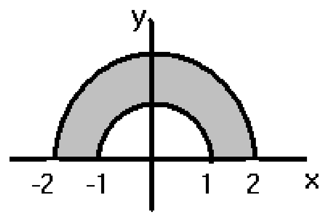
$$\begin{aligned} \int_D \left(x^2+y^2\right)dA &= \int_0^{\frac{10}{3}}\int_0^{\frac{10-3x}{4}} \left(x^2+y^2\right)dy\,dx \\ &= \int_0^{\frac{10}{3}} \left(x^2y+\frac{1}{3}y^3\right)\Big|_0^{\frac{10-3x}{4}} \\ &= \int_0^{\frac{10}{3}} \left(x^2\left(\frac{10-3x}{4}\right)+\frac{1}{3}\left(\frac{10-3x}{4}\right)^3\right) \\ &= \left(\frac{10}{12}x^3-\frac{3}{16}x^4+\frac{1}{3}\left(\frac{1}{4}\left(\frac{10-3x}{4}\right)\left(-\frac{4}{3}\right)\right)\right)\Big|_0^{\frac{10}{3}} \\ &= \left(\frac{10}{12}\left(\frac{10}{3}\right)^3-\frac{3}{16}\left(\frac{10}{3}\right)^4+0-\left(\frac{1}{3}\right)\left(\frac{1}{4}\right)\left(\frac{10}{4}\right)\left(-\frac{4}{3}\right)\right) \end{aligned}$$

□

5.3.4. Let $D = \{(x,y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 2 \quad , \quad y \geq 0\}$. Is D an elementary region? Evaluate

$$\int_D (1+xy)\, dA \quad .$$

- A sketch of the region D appears below



To evaluate the integral over D, we regard D as the union of three regions of type I

$$D = D_1 \cup D_2 \cup D_3$$

$$\begin{aligned}
D_1 &= \left\{(x,y) \in \mathbb{R}^2 \mid -\sqrt{2} \leq x \leq -1 \quad , \quad 0 \leq y \leq \sqrt{2-x^2}\right\} \\
D_2 &= \left\{(x,y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1 \quad , \quad \sqrt{1-x^2} \leq y \leq \sqrt{2-x^2}\right\} \\
D_1 &= \left\{(x,y) \in \mathbb{R}^2 \mid 1 \leq x \leq \sqrt{2} \quad , \quad 0 \leq y \leq \sqrt{2-x^2}\right\}
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_D (1+xy) dA &= \int_{D_1} (1+xy) dA + \int_{D_2} (1+xy) dA + \int_{D_3} (1+xy) dA \\
&= \int_{-\sqrt{2}}^{-1} \int_0^{\sqrt{2-x^2}} (1+xy)\, dy\, dx + \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{2-x^2}} (1+xy)\, dy dx \\
&\qquad \int_1^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} (1+xy)\, dy\, dy
\end{aligned}$$

$$\begin{aligned}
&= \int_{-\sqrt{2}}^{-1} \left(y + \frac{1}{2}xy^2\right)\Bigg|_0^{\sqrt{2-x^2}} dx + \int_{-1}^{1} \left(y + \frac{1}{2}xy^2\right)\Bigg|_{\sqrt{1-x^2}}^{\sqrt{2-x^2}} dx \\
&\qquad + \int_1^{\sqrt{2}} \left(y + \frac{1}{2}xy^2\right)\Bigg|_0^{\sqrt{2-x^2}} dx \\
&= \int_{-\sqrt{2}}^{-1} \left(\sqrt{2-x^2} + \frac{1}{2}x\left(2-x^2\right)\right) dx + \int_{-1}^{1} \left(\sqrt{2-x^2} + \frac{1}{2}x\left(2-x^2\right)\right) dx \\
&\qquad - \int_{-1}^{1} \left(\sqrt{1-x^2} + \frac{1}{2}x\left(1-x^2\right)\right) + \int_1^{\sqrt{2}} \left(\sqrt{2-x^2} + \frac{1}{2}x\left(2-x^2\right)\right) dx \\
&= \int_{-\sqrt{2}}^{\sqrt{2}} \left(\sqrt{2-x^2} + \frac{1}{2}\left(2x-x^3\right)\right) dx - \int_{-1}^{1} \left(\sqrt{1-x^2} + \frac{1}{2}\left(x-x^3\right)\right) dx
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{x}{2}\sqrt{2-x^2}+\frac{2}{2}\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\right)\Bigg|_{-\sqrt{2}}^{\sqrt{2}} - \left(\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}(x)\right)\Bigg|_{-1}^{1} \\
&= \left(0+\sin^{-1}(1)-0-\sin^{-1}(-1)-0-\frac{1}{2}\sin^{-1}(1)+0-\frac{1}{2}\sin^{-1}(-1)\right) \\
&= \frac{\pi}{2}+\frac{\pi}{2}-\frac{\pi}{4}-\frac{\pi}{4} \\
&= \frac{\pi}{2}
\end{aligned}
$$

□

Section 5.4

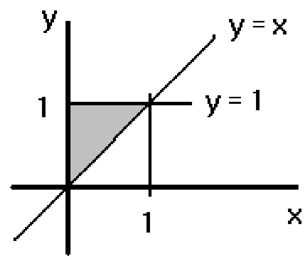
5.4.1(a). Change the order of integration, sketch the corresponding region, and evaluate the following integrals both ways.

(a)

$$\int_0^1 \int_x^1 (xy) dy\, dx$$

- Noting the limits of integration, we see that this iterated integral corresponds to the type I

$$D_I = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 1 \quad , \quad x \le y \le 1\right\}$$



From the sketch above, we see that region can also be regarded as the type II

$$D_{II} = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \le y \le 1 \quad , \quad 0 \le x \le y\right\} \quad .$$

We thus have

$$\int_0^1 \int_x^1 (xy) dy\, dx = \int_{D_I} (xy) dA = \int_{D_{II}} (xy) dA = \int_0^1 \int_0^y (xy)\, dx dy$$

Computing the iterated integral on the left hand side, we obtain

$$\begin{aligned} \int_0^1 \int_x^1 (xy) dy\, dx &= \int_0^1 \left(\frac{1}{2}xy^2\right)\Big|_x^1 dx \\ &= \int_0^1 \left(\frac{1}{2}x - \frac{1}{2}x^3\right) dx \\ &= \left(\frac{1}{4}x^2 - \frac{1}{8}x^4\right)\Big|_0^1 \\ &= \frac{1}{4} - \frac{1}{8} \\ &= \frac{1}{8} \end{aligned}$$

Computing the iterated integral on the far right hand side, we obtain

$$\begin{aligned} \int_0^1 \int_0^y (xy) dx\, dy &= \int_0^1 \left(\frac{1}{2}xy^2\right)\Big|_0^y dy \\ &= \int_0^1 \left(\frac{1}{2}y^3\right) dy \\ &= \left(\frac{1}{8}y^4\right)\Big|_0^1 \\ &= \frac{1}{8} \end{aligned}$$

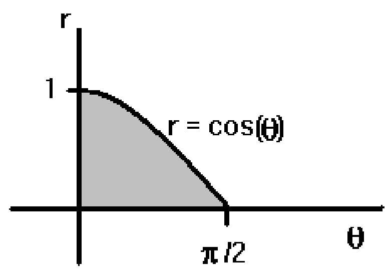
□

(b)

$$\int_0^{\frac{\pi}{2}} \int_0^{\cos(\theta)} \cos(\theta) dr\, d\theta \quad .$$

- Noting the limits of integration, we see that this iterated integral corresponds to the type II

$$D_I = \left\{ (r, \theta) \in \mathbb{R}^2 \mid 0 \le \theta \le \frac{\pi}{2} \quad , \quad 0 \le r \le \cos(\theta) \right\}$$



From the sketch above, we see that region can also be regarded as the type II

$$D_{II} = \left\{ (r, \theta) \in \mathbb{R}^2 \mid 0 \le r \le 1 \quad , \quad 0 \le \theta \le \cos^{-1}(r) \right\} \quad .$$

We thus have

$$\int_0^{\frac{\pi}{2}} \int_0^{\cos(\theta)} \cos(\theta) dr\, d\theta = \int_{D_I} \cos(\theta) dA = \int_{D_{II}} \cos(\theta) dA = \int_0^1 \int_0^{\cos^{-1}(r)} \cos(\theta) d\theta\, dr$$

Computing the iterated integral on the far left hand side yields

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \int_0^{\cos(\theta)} \cos(\theta) dr\, d\theta &= \int_0^{\frac{\pi}{2}} r\cos(\theta) \Big|_0^{\cos(\theta)} d\theta \\ &= \int_0^{\frac{\pi}{2}} \cos^2(\theta) d\theta \\ &= \left(\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) \right) \Big|_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{4} \end{aligned}$$

Computing the iterated integral on the far right hand side yields

$$\begin{aligned} \int_0^1 \int_0^{\cos^{-1}(r)} \cos(\theta) d\theta\, dr &= \int_0^1 \sin(\theta) \Big|_0^{\cos^{-1}(r)} dr \\ &= \int_0^1 \sin\left(\cos^{-1}(r)\right) dr \end{aligned}$$

To carry out this last integration we make a change of variables

$$\begin{aligned} r &= \cos(u) \\ dr &= -\sin(u) du \\ r = 1 &\Leftrightarrow \quad u = 0 \\ r = 0 &\Leftrightarrow \quad u = \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned}\int_0^1 \sin\left(\cos^{-1} * r\right)\right) dr &= \int_{\frac{\pi}{2}}^0 \sin(u)\left(-\sin(u)du\right) \\ &= \int_0^{\frac{\pi}{2}} \sin^2(u)du \\ &= \left(\frac{1}{2}u - \frac{1}{4}\sin(2u)\right)\Big|_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{4}\end{aligned}$$

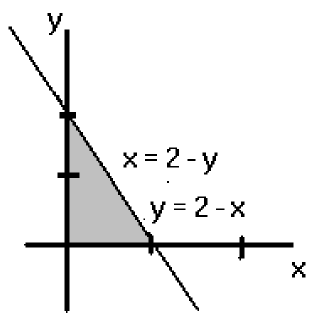
□

(c)

$$\int_0^1 \int_1^{2-y} (x+y)^2 \, dxdy$$

- Noting the limits of integration, we see that this iterated integral corresponds to the type II

$$D_{II} = \left\{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 1 \quad , \quad 1 \le x \le 2-y\right\}$$



From the sketch above, we see that region can also be regarded as the type I

$$D_I = \left\{(x,y) \in \mathbb{R}^2 \mid 1 \le x \le 2 \quad , \quad 0 \le y \le 2-x\right\} \quad .$$

We thus have

$$\int_0^1 \int_1^{2-y} (x+y)^2 dx\, dy = \int_{D_{II}} (x+y)^2 dA = \int_{D_I} (x+y)^2 dA = \int_1^2 \int_0^{2-x} (x+y)^2 dy dx$$

Computing the iterated integral on the far left hand side, we obtain

$$\begin{aligned}\int_0^1 \int_1^{2-y} (x+y)^2 dx\, dy &= \int_0^1 \int_1^{2-y} \frac{1}{3}(x+y)^3\Big|_1^{2-y} dx\, dy \\ &= \int_0^1 \left(\frac{1}{3}\left(2^3 - (y+1)^3\right)\right) dy \\ &= \left(\frac{8}{3}y - \frac{1}{12}(y+1)^4\right)\Big|_0^1 \\ &= \frac{8}{3} - \frac{16}{12} - 0 + \frac{1}{12} \\ &= \frac{17}{12}\end{aligned}$$

Computing the iterated integral on the far left hand side, we obtain

$$\begin{aligned}\int_1^2 \int_0^{2-x} (x+y)^2 dy dx &= \int_1^2 \frac{1}{3}(x+y)^3\Big|_0^{2-x} dx \\ &= \int_1^2 \frac{1}{3}\left(2^3 - x^3\right) dx \\ &= \frac{1}{3}\left(8x - \frac{1}{4}x^4\right)\Big|_1^2 \\ &= \frac{16}{3} - \frac{16}{12} - \frac{8}{3} + \frac{1}{12} \\ &= \frac{17}{12}\end{aligned}$$

□

5.4.2. Compute the volume of the ellipsoid with semiaxes a, b, and c. (Hint: use symmetry and first find the volume of half the ellipsoid.)

- The boundary of the ellipsoid with semiaxes a, b, and c is the solution set of the equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1 \quad .$$

To compute the volume of this solid, we first observe that the top half of this solid is just the volume under the graph of

$$z = f(x,y) = \sqrt{c^2 - \left(\frac{cx}{a}\right)^2 - \left(\frac{cy}{b}\right)^2} \quad .$$

lying above the region

$$R = \left\{(x,y) \in \mathbb{R}^2 \mid \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \le 1 \quad .\right\}$$

in the xy-plane. Thus, the volume of the ellipsoid should be twice that of volume lying under the graph of $f(x,y)$ and above the region R. Thus,

$$Vol = 2\int_R f(x,y) dA \quad .$$

Now we can regard the region R as the type I region prescribed by

$$R = \left\{(x,y) \in \mathbb{R}^2 \mid -a \le x \le a \quad , \quad -\sqrt{b^2 - \left(\frac{bx}{a}\right)^2} \le y \le \sqrt{b^2 - \left(\frac{bx}{a}\right)^2}\right\} \quad .$$

Thus,

$$\begin{aligned}Vol &= 2\int_{-a}^{a}\int_{-\sqrt{b^2-\left(\frac{bx}{a}\right)^2}}^{\sqrt{b^2-\left(\frac{bx}{a}\right)^2}} \sqrt{c^2 - \left(\frac{cx}{a}\right)^2 - \left(\frac{cy}{b}\right)^2}\, dy\, dx \\ &= 2c\int_{-a}^{a}\int_{-\sqrt{b^2-\left(\frac{bx}{a}\right)^2}}^{\sqrt{b^2-\left(\frac{bx}{a}\right)^2}} \frac{1}{b}\sqrt{b^2 - \left(\frac{bx}{a}\right)^2 - y^2}\, dy\, dx\end{aligned}$$

$$= 2c\int_{-a}^{a} \frac{1}{b}\left(\frac{y}{2}\sqrt{b^2 - \left(\frac{bx}{a}\right)^2 - y^2} + \frac{b^2 - \left(\frac{bx}{a}\right)^2}{2}\sin^{-1}\left(\frac{y}{\sqrt{b^2 - \left(\frac{bx}{a}\right)^2}}\right)\right)\Bigg|_{-\sqrt{b^2-\left(\frac{bx}{a}\right)^2}}^{\sqrt{b^2-\left(\frac{bx}{a}\right)^2}} dx$$

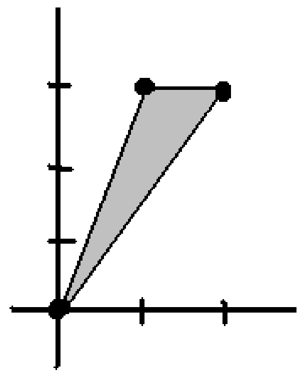


FIGURE 1

$$
\begin{aligned}
&= 2c\int_{-a}^{a}\frac{1}{b}\left(0+\frac{b^2-\left(\frac{bx}{a}\right)^2}{2}\sin^{-1}(1)-0-\frac{b^2-\left(\frac{bx}{a}\right)^2}{2}\sin^{-1}(-1)\right)dx \\
&= bc\int_{-a}^{a}\left(1+\left(\frac{x}{a}\right)^2\right)\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)dx \\
&= \pi bc\int_{-a}^{a}\left(1-\frac{x^2}{a^2}\right)dx \\
&= \pi bc\left(x-\frac{1}{3}\frac{x^3}{a^2}\right)\Big|_{-a}^{a} \\
&= \pi bc\left(a-\frac{1}{3}a-\left(-a-\frac{1}{3}a\right)\right) \\
&= \frac{4}{3}\pi abc
\end{aligned}
$$

□

5.4.3. Evaluate

$$\int_D e^{x-y}\,dA$$

where D is the interior of the triangle with vertices (0,0), (1,3) and (2,2).

- This regions looks like Observe the region D is bounded by the lines

$$
\begin{aligned}
y &= 3x \quad , \quad 0\le x\le 1 \\
y &= 4-x \quad , \quad 1\le x\le 2 \\
y &= x \quad , \quad 0\le x\le 2
\end{aligned}
$$

and can thus be regarded as the union of two type I regions

$$D = D_1\cup D_2$$

where

$$
\begin{aligned}
D_1 &= \left\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 1 \quad , \quad x\le y\le 3x\right\} \\
D_2 &= \left\{(x,y)\in\mathbb{R}^2 \mid 1\le x\le 2 \quad , \quad x\le y\le 4-x\right\} \quad .
\end{aligned}
$$

We thus have

$$
\begin{aligned}
\int_D e^{x-y} dA &= \int_{D_1} e^{x-y} dA + \int_{D_2} e^{x-y} dA \\
&= \int_0^1 \int_x^{3x} e^{x-y} dy\, dx + \int_1^2 \int_x^{4-x} e^{x-y} dy\, dx \\
&= \int_0^1 \left(-e^{x-y}\right)\Big|_x^{3x} dx + \int_1^2 \left(-e^{x-y}\right)\Big|_x^{4-x} dx \\
&= \int_0^1 \left(-e^{-2x} + 1\right) dx + \int_1^2 \left(-e^{2x-4} + 1\right) dx \\
&= \left(\frac{1}{2}e^{-2x} + x\right)\Big|_0^1 + \left(-\frac{1}{2}e^{2x-4} + x\right)\Big|_1^2 \\
&= \frac{1}{2}e^{-2} + 1 - \frac{1}{2} - 0 - \frac{1}{2}e^0 + 2 + \frac{1}{2}e^{-2} - 1 \\
&= e^{-2} + 1
\end{aligned}
$$

□

5.4.4. Evaluate

$$
\int_D y^3 \left(x^2 + y^2\right)^{-3/2} dA
$$

where D is the region determined by the conditions $\frac{1}{2} \leq y \leq 1$ and $x^2 + y^2 \leq 1$.

- D is just the portion of the unit disk that lies above the line $y = 1$. This region is easily rendered as region of Type I: noting that the line $y = \frac{1}{2}$ intersects the unit circle at $\left(\frac{-\sqrt{3}}{2}, \frac{1}{2}\right)$ and $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, we see that

$$
D = \left\{ (x,y) \in \mathbb{R}^2 \mid -\frac{\sqrt{3}}{2} \leq x \leq \frac{\sqrt{3}}{2} \quad , \quad \frac{1}{2} \leq y \leq \sqrt{1 - x^2} \right\} \quad .
$$

So

$$
\int_D y^3 \left(x^2 + y^2\right)^{-3/2} dA = \int_{-\sqrt{3}/2}^{\sqrt{3}/2} dx \int_{\frac{1}{2}}^{\sqrt{1-x^2}} dy \left(y^2 \left(x^2 + y^2\right)^{-\frac{3}{2}}\right)
$$

The first integral looks a little difficult. Let's see if things simplify when we regard D as a region of Type II. If we prescribe D by

$$
D = \left\{ (x,y) \in \mathbb{R}^2 \mid \frac{1}{2} \leq y \leq 1 \quad , \quad -\sqrt{1 - y^2} \leq x \leq \sqrt{1 - y^2} \right\}
$$

then

$$\begin{aligned}
\int_D y^3 \left(x^2+y^2\right)^{-3/2} dA &= \int_{\frac{1}{2}}^{1} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y^3 \left(x^2+y^2\right)^{-\frac{3}{2}} dx \\
&= \int_{\frac{1}{2}}^{1} \left[y^3 \left(\frac{x}{y^2\sqrt{x^2+y^2}} \right) \Big|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \right] dy \\
&= \int_{\frac{1}{2}}^{1} 2y\sqrt{1-y^2} dy \\
&= -\frac{2}{3}\left(1-y^2\right)^{3/2} \Big|_{\frac{1}{2}}^{1} \\
&= \frac{2}{3}\left(\frac{3}{4}\right)^{3/2}
\end{aligned}$$

□

Section 5.6

5.6.1. Evaluate

$$\int_W x^2 dV$$

where $W = [0,1] \times [0,1] \times [0,1]$.

- We have

$$\begin{aligned}
\int_W x^2 dV &= \int_0^1 \int_0^1 \int_0^1 x^2 \, dz \, dy \, dx \\
&= \int_0^1 \int_0^1 x^2 z\big|_0^1 \, dy \, dx \\
&= \int_0^1 \int_0^1 x^2 dy \, dx \\
&= \int_0^1 x^2 y\big|_0^1 \, dx \\
&= \int_0^1 x^2 dx \\
&= \frac{1}{3}x^3 \Big|_0^1 \\
&= \frac{1}{3}
\end{aligned}$$

□

5.6.2. Evaluate

$$\int_W ye^{-xy} dV$$

where $W = [0,1] \times [0,1] \times [0,1]$.

-

$$\begin{aligned}
\int_W y e^{-xy} dV &= \int_0^1 \int_0^1 \int_0^1 y e^{-xy} dz\, dx\, dy \\
&= \int_0^1 \int_0^1 y e^{-xy} z \Big|_0^1 dx\, dy \\
&= \int_0^1 \int_0^1 y e^{-xy} dx\, dy \\
&= \int_0^1 y \left(-\frac{1}{y} e^{-xy} \right) \Big|_0^1 dy \\
&= \int_0^1 \left(-e^{-x} + 1 \right) dx \\
&= \left(e^{-x} + x \right) \Big|_0^1 \\
&= e^{-1}
\end{aligned}$$

□

5.6.3. Evaluate

$$\int_W (2x+3y+z)dV$$

where $W = [1,2]\times[-1,1]\times[0,1]$.

- $$\begin{aligned}
\int_W (2x+3y+z)dV &= \int_1^2\int_{-1}^1\int_0^1 (2x+3y+z)dz\,dy\,dx \\
&= \int_1^2\int_{-1}^1 \left(2xz+3yz+\frac{1}{2}z^2\right)\Big|_0^1 dy\,dx \\
&= \int_1^2\int_{-1}^1 \left(2x+3y+\frac{1}{2}\right) dy\,dx \\
&= \int_1^2 \left(2xy+\frac{3}{2}y^2+\frac{1}{2}y\right)\Big|_{-1}^1 dx \\
&= \int_1^2 \left(2x+\frac{3}{2}+\frac{1}{2}-\left(-2x+\frac{3}{2}-\frac{1}{2}\right)\right) dx \\
&= \int_1^2 (4x+1)\,dx \\
&= \left(2x^2+x\right)\Big|_1^2 \\
&= 8+2-2-1 \\
&= 7
\end{aligned}$$

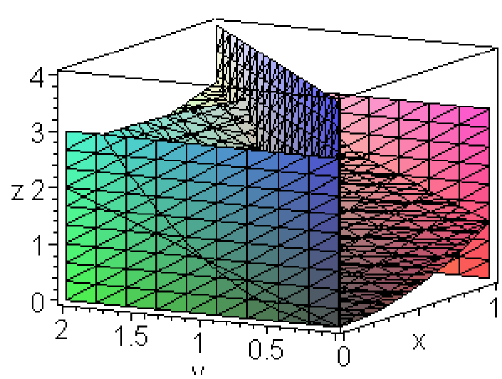
□

5.6.4. Evaluate

$$\int_0^1\int_0^{2x}\int_{x^2+y^2}^{x+y} dz\,dy\,dx$$

and sketch the region of integration.

- The region of integration is bounded by the planes $x=0$, $x=1$, $y=0$, $y=2x$, $z=x+y$ and the surface $z=x^2+y^2$.



$$\begin{aligned}
\int_0^1 \int_0^{2x} \int_{x^2+y^2}^{x+y} dz\, dy\, dx &= \int_0^1 \int_0^{2x} z \,\Big|_{x^2+y^2}^{x+y} dy\, dx \\
&= \int_0^1 \int_0^{2x} \left(x + y - x^2 - y^2\right) dy \; dx \\
&= \int_0^1 \left(xy + \frac{1}{2}y^2 - x^2 y - \frac{1}{3}y^3 \right)\Big|_0^{2x} dx \\
&= \int_0^1 \left(2x^2 + \frac{1}{2}(4x^2) - 2x^3 - \frac{8}{3}x^3 \right) dx \\
&= \int_0^1 \left(4x^2 - \frac{14}{3}x^3 \right) dx \\
&= \left(\frac{4}{3}x^3 - \frac{14}{12}x^4 \right)\Big|_0^1 \\
&= \frac{4}{3} - \frac{7}{6} \\
&= \frac{1}{6}
\end{aligned}$$

□

5.6.6. Compute the integral of the function $f(x, y, z) = z$ over the region W in the first octant of $\mathbb{R}^3$ bounded by the planes $y = 0$, $z = 0$, $x + y = 2$, $2y + x = 6$, and the cylinder $y^2 + z^2 = 4$.

- We first need to develop a description of the region W that will allow us to determine the appropriate limits of integration for an iterated integral over W.

  We first consider the range of the variable z for fixed x and y. From the equations of the boundaries we see that

  $$0 \leq z \leq \sqrt{4 - y^2} \quad .$$

  Let us first consider the maximal range of y for arbitray x and z. The boundary equation

  $$x + y = 2$$

  constrains y to be less than y since x (and y and z) must be positive. The boundary equation

  $$2y + x = 6$$

  is a bit weaker, in merely requires $y \leq 3$. The last boundary equation

  $$y^2 + z^2 = 4$$

  constrains y also constrains $y \leq 0$. We conclude that the variable y ranges from 0 to 2 over the region of integration.

  We next consider the range of x for fixed y. We have two boundary equations relating y to x. They are equivalent to

  $$\begin{aligned} x &= 2 - y \\ x &= 6 - 2y \end{aligned}$$

  Noting that if $y \in [0, 2]$

  $$2 - y < 6 - 2y \quad ,$$

  we conclude that the range of x for fixed y is given by

  $$2 - y \leq x \leq 6 - 2y \quad .$$

  Finally, we consider the range of z. In view of the requirements

  $$\begin{aligned} z &\geq 0 \\ y^2 + z^2 &\leq 4 \end{aligned}$$

we conclude

$$0 \leq z \leq \sqrt{4-y^2}.$$

Thus,

$$W = \left\{ (x,y,z) \in \mathbb{R}^3 \mid 0 \leq y \leq 2 \quad , \quad 2-y \leq x \leq 6-2y \quad , \quad 0 \leq z \leq \sqrt{4-y^2} \right\}$$

and

$$\begin{aligned} \int_W f(x,y,z)dV &= \int_0^2 \int_{2-y}^{6-2y} \int_0^{\sqrt{4-y^2}} z\,dz\,dx\,dy \\ &= \int_0^2 \int_{2-y}^{6-2y} \frac{1}{2}\left(4-y^2\right) dx\,dy \\ &= \frac{1}{2}\int_0^2 \left(4-y^2\right)(6-2y-2+y)\,dy \\ &= \frac{1}{2}\int_0^2 \left(4-y^2\right)(4-y)\,dy \\ &= \frac{1}{2}\int_0^2 \left(16-4y-4y^2+y^3\right) \\ &= \frac{1}{2}\left(16y-2y^2-\frac{4}{3}y^3+\frac{1}{4}y^4\right)\Big|_0^2 \\ &= 16-4-\frac{16}{3}+2 \\ &= \frac{26}{3} \end{aligned}$$

□

5.6.7. Evaluate

$$\int_S xyz\,dV$$

where S is the region determined by the conditions $x \geq 0$, $y \geq 0$, $z \geq 0$, and $x^2+y^2+z^2 \leq 1$.

- For fixed x and y we have

$$0 \leq z \leq \sqrt{1-x^2-y^2}$$

For fixed x we have y restricted to lie between

$$0 \leq y \leq \sqrt{1-x^2}$$

and then finally, x is restricted to lie between

$$0 \leq x \leq 1$$

Thus, S is an elementary region and the integral over S is calculable as

$$\begin{aligned}
\int_S xyz\,dV &= \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\;dz\;dy\;dx \\
&= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{1}{2}xyz^2\Big|_0^{\sqrt{1-x^2-y^2}}\;dy\;dx \\
&= \frac{1}{2}\int_0^1 \int_0^{\sqrt{1-x^2}} xy\left(1-x^2-y^2\right)dy\;dx \\
&= \frac{1}{2}\int_0^1 \left(\frac{1}{2}xy^2-\frac{1}{2}x^3y^2-\frac{1}{4}xy^4\right)\Big|_0^{\sqrt{1-x^2}}\;dx \\
&= \frac{1}{2}\int_0^1 \left(\frac{1}{2}x(1-x^2)-\frac{1}{2}x^3(1-x^2)-\frac{1}{4}x(1-x^2)^2\right)dx \\
&= \frac{1}{8}\int_0^1 \left(2x-2x^3-2x^3+2x^5-x+2x^3-x^5\right)dx \\
&= \frac{1}{8}\int_0^1 \left(x-2x^3+x^5\right)dx \\
&= \frac{1}{8}\left(\frac{1}{2}x^2-\frac{1}{2}x^4+\frac{1}{6}x^6\right)\Big|_0^1 \\
&= \frac{1}{8}\left(\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right) \\
&= \frac{1}{48}
\end{aligned}$$

□