Math 4013
Solutions to Homework Problems from Chapter 4

Section 4.2

4.2.1. Calculate the arc length of the following curves.

(a) $\sigma(t) = \left(6t, 3t^2, t^3\right) \quad , \quad t \in [0,1]$

- Well,

$$\sigma'(t) = \left(6, 6t, 3t^2\right)$$

so,

$$\begin{aligned} \|\sigma'(t)\| &= \sqrt{36 + 36t^2 + 9t^4} \\ &= \sqrt{9\left(t^4 + 4t^2 + 4\right)} \\ &= \sqrt{9\left(t^2 + 2\right)^2} \\ &= 3\left(t^2 + 2\right) \end{aligned}$$

Thus,

$$\begin{aligned} L[\sigma] &= \int_{t_i}^{t_f} \|\sigma'(t)\|\, dt \\ &= \int_0^1 3\left(t^2 + 2\right) dt \\ &= \left(t^3 + 6t\right)\Big|_0^1 \\ &= 7 \end{aligned}$$

∎

(b) $\sigma(t) = \left(\sin(3t), \cos(3t), 2t^{3/2}\right) \quad , \quad t \in [0,1]$

- Well,

$$\sigma'(t) = \left(3\cos(3t), -3\sin(3t), 3t^{1/2}\right)$$

so,

$$\begin{aligned} \|\sigma'(t)\| &= \sqrt{9\cos^2(3t) + 9\sin^2(3t) + 9t} \\ &= \sqrt{9\left(1 + t\right)} \\ &= 3\sqrt{1 + t} \end{aligned}$$

Thus,

$$\begin{aligned} L[\sigma] &= \int_{t_i}^{t_f} \|\sigma'(t)\|\, dt \\ &= \int_0^1 3\sqrt{1+t}\,dt \\ &= \int_1^2 3\sqrt{u}\,du \qquad , \qquad u = 1 + t \\ &= u^{3/2}\Big|_1^2 \\ &= 2^{3/2} - 1 \end{aligned}$$

∎

4.2.2. Let σ be the path $\sigma(t) = (t, t\sin(t), t\cos(t))$. Find the arc length of σ between (0,0,0) and $(\pi, 0, -\pi)$.

- Well,

$$\sigma'(t) = (1, \sin(t) + t\cos(t), \cos(t) - t\sin(t))$$

so

$$\begin{aligned} \|\sigma'(t)\| &= \sqrt{1^2 + (\sin(t) + t\cos(t))^2 + (\cos(t) - t\sin(t))^2} \\ &= \sqrt{1 + \sin^2(t) + t^2\cos^2(t) + \cos^2(t) + t^2\sin^2(t)} \\ &= \sqrt{2 + t^2} \end{aligned}$$

Note also that we must have $t_i = 0$ and $t_f = \pi$ so that

$$\begin{aligned} \sigma(t_i) &= (0, 0, 0) \\ \sigma(t_f) &= (\pi, 0, -\pi) \end{aligned}$$

Therefore, the arc lenght will be given by the following integral

$$\begin{aligned} L[\sigma] &= \int_{t_i}^{t_f} \|\sigma'(t)\|\, dt \\ &= \int_0^{\pi} \sqrt{2 + t^2}\, dt \\ &= \frac{t}{2}\sqrt{t^2 + 2} + \frac{2}{2}\log\left|t + \sqrt{t^2 + 2}\right|\Big|_0^{\pi} \\ &= \frac{\pi}{2}\sqrt{\pi^2 + 2} + \log\left|\pi + \sqrt{\pi^2 + 2}\right| - \log\left|\sqrt{2}\right| \end{aligned}$$

(See integral #43 in the tables at the back of the text.) ∎

Section 4.3

4.3.1. A particle of mass m moves along a path $\mathbf{r}(t)$ according to Newton's law in a force field $\mathbf{F} = -\nabla V$ on $\mathbb{R}^3$, where V is a given potential energy function.

(a) Prove that in the energy along the trajectory

$$E = \frac{1}{2}m\|\mathbf{r}'(t)\|^2 + V(\mathbf{r}(t))$$

is constant in time.

- We have

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt}\left(\frac{1}{2}m\|\mathbf{r}'(t)\|^2 + V(\mathbf{r}(t))\right) \\ &= \frac{m}{2}\frac{d}{dt}(\mathbf{r}'(t)\cdot\mathbf{r}'(t)) + \frac{d}{dt}(V(\mathbf{r}(t))) \\ &= \frac{m}{2}(\mathbf{r}''(t)\cdot\mathbf{r}'(t) + \mathbf{r}'(t)\cdot\mathbf{r}''(t)) + \nabla V\cdot\frac{d\mathbf{r}}{dt} \\ &= m\mathbf{r}''(t)\cdot\mathbf{r}'(t) + \nabla V\cdot\mathbf{r}'(t) \end{aligned}$$

(In the third line we have simply applied the product and chain rules to, respectively, the first and second terms of the second line.) According to Newton's law $\mathbf{F} = m\mathbf{a}$, so

$$m\mathbf{r}'' = \mathbf{F} = -\nabla V \quad .$$

Thus,

$$\frac{dE}{dt} = -\nabla V\cdot\mathbf{r}'(t) + \nabla V\cdot\mathbf{r}'(t) = 0 \quad .$$

∎

(b) If the particle moves on an equipotential surface, show that its speed is constant.

- Well, the particle speed is just the magnitude of the velocity vector. So it suffices to prove that

$$\frac{d}{dt}\left(\|\mathbf{r}'(t)\|^2\right) = 0$$

whenever the particle moves along an equipotential surface.

But

$$\begin{aligned} \frac{d}{dt}\left(\|\mathbf{r}'(t)\|^2\right) &= \frac{d}{dt}\left(\mathbf{r}'(t)\cdot\mathbf{r}'(t)\right) \\ &= \mathbf{r}''(t)\cdot\mathbf{r}'(t) + \mathbf{r}'(t)\cdot\mathbf{r}''(t) \\ &= 2\mathbf{r}'(t)\cdot\mathbf{r}''(t) \\ &= \frac{2}{m}\mathbf{r}'(t)\cdot\left(m\mathbf{r}''(t)\right) \\ &= \frac{-2}{m}\mathbf{r}'(t)\cdot\nabla V\left(\mathbf{r}(t)\right) \end{aligned}$$

Now we know from Section 2.5, that the gradient vector ∇V evaluated at $\mathbf{r}(t)$ will be normal to the surface

$$S = \left\{\mathbf{x}\in\mathbb{R}^3 \mid V(\mathbf{x}) = k\right\}$$

at the point $\mathbf{r(t)}$. On the other hand, since the trajectory is constrained to lie in such a surface, the tangent vector $\mathbf{r}'(t)$ at a point $\mathbf{r}(t)$ must always be perpendicular to the surface normal. In other words,

$$\mathbf{r}'(t)\cdot\nabla V\left(\mathbf{r}(t)\right) = 0 \quad .$$

Thus,

$$\frac{d}{dt}\left(\|\mathbf{r}'(t)\|^2\right) = -\frac{2}{m}\mathbf{r}'(t)\cdot\nabla V\left(\mathbf{r}(t)\right) = 0 \quad .$$

∎

4.3.2. Sketch a few flow lines of the vector field $\mathbf{F}(x,y) = (x,-y)$.

- The flow lines for this vector field must satisfy the differential equation

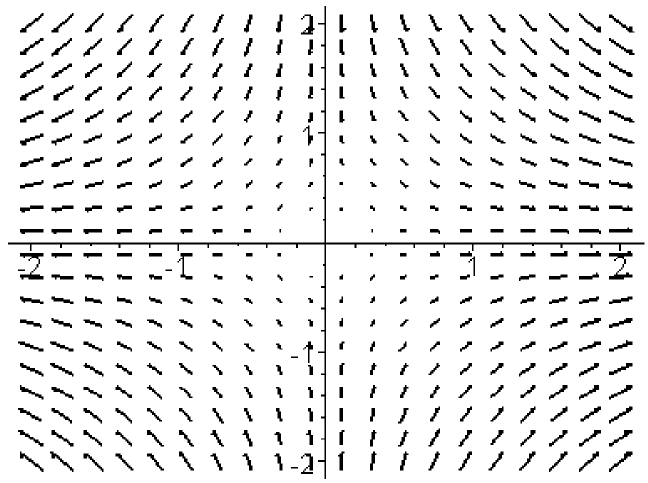
$$\frac{d\sigma}{dt} = \mathbf{F}\left(\sigma(t)\right)$$

But

$$\begin{pmatrix} \frac{d\sigma_x}{dt} \\ \frac{d\sigma_y}{dt} \end{pmatrix} = \begin{pmatrix} \sigma_x(t) \\ -\sigma_y(t) \end{pmatrix} \quad \Rightarrow \quad \begin{array}{rclcl} \frac{d\sigma_x}{dt} &=& \sigma_x &\Rightarrow& \sigma_x(t) = x_o e^t \\ \frac{d\sigma_y}{dt} &=& -\sigma_y &\Rightarrow& \sigma_y(t) = y_o e^{-t} \end{array}$$

so the flow lines of $\mathbf{F}$ will be curves of the form

$$\sigma(t) = \left(x_o e^t, y_o e^{-t}\right) \quad .$$



∎

4.3.3. Let $\mathbf{c}(t)$ be a flow line of a gradient field $\mathbf{F} = -\nabla V$. Prove that $V(\mathbf{c}(t))$ is a decreasing function of t. Explain.

•

$$\begin{aligned} \frac{d}{dt}\left[V(\mathbf{c}(t))\right] &= \nabla V(\mathbf{c}(t)) \cdot \frac{d\mathbf{c}}{dt} \\ &= \nabla V(\mathbf{c}(t)) \cdot \mathbf{F}(\mathbf{c}(t)) \\ &= \nabla V(\mathbf{c}(t)) \cdot (-\nabla V(\mathbf{c}(t))) \\ &= -\|\nabla V(\mathbf{c}(t))\|^2 \end{aligned}$$

Since the magnitude of a vector is either positive or zero, we conclude that $\frac{d}{dt}\left[V(\mathbf{c}(t))\right]$ is either negative or zero.

To understand this, recall that $-\nabla V(\mathbf{r})$ represents the direction of the fastest decrease in V at the point $\mathbf{r}$. Thus, the the flow lines of a vector field $\mathbf{F} = -\nabla V$ will always move in the direction of the fastest decrease in V; V obviously V will be decreasing along these flow lines.

In a physical situation, $\mathbf{F}$ is interpretable as a force field and V is a corresponding potential energy. The fact that V is always decreasing along the flow lines of $\mathbf{F} = -\nabla V$ implies that a particle acted upon by $\mathbf{F}$ always moves along a path that decreases its potential energy. (Now you know why apples fall.) ∎

4.3.4. Sketch the gradient field $-\nabla V$ for $V(x,y) = (x+y)/\left(x^2+y^2\right)$. Sketch the equipotential surface $V = 1$.

- The easiest way to approach this problem is first uncover the nature of the equipotential surfaces. Now the points on an equipotential surface for V must satisfy an equation of the form

$$\frac{x+y}{x^2+y^2} = k$$

which is equivalent to

$$x^2 - \frac{1}{k}x + y^2 - \frac{1}{k}y = 0$$

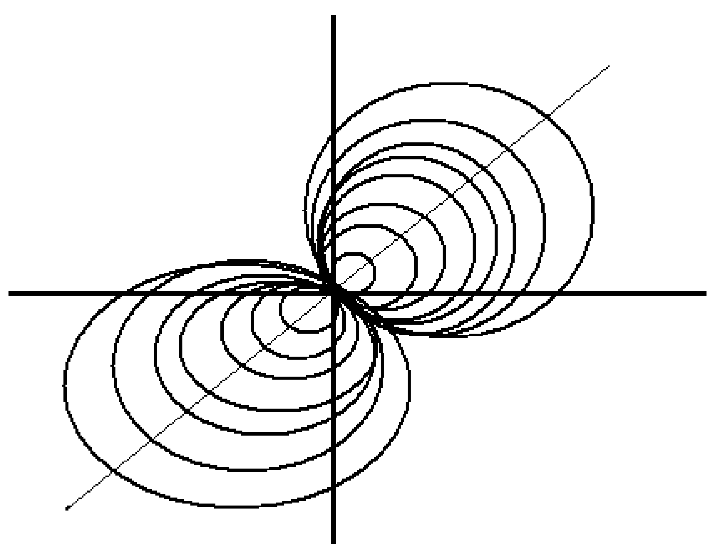
which, upon adding $2\left(\frac{1}{2k}\right)^2$ to both sides, becomes

$$x^2 - \frac{1}{k}x + \left(\frac{1}{2k}\right)^2 + y^2 - \frac{1}{k}y + \left(\frac{1}{2k}\right)^2 = 2\left(\frac{1}{2k}\right)^2$$

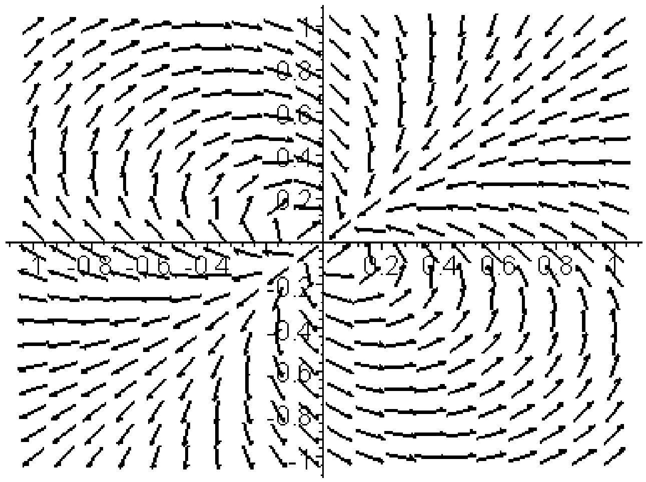
or

$$\left(x - \frac{1}{2k}\right)^2 + \left(y - \frac{1}{2k}\right)^2 = \frac{1}{2k^2}.$$

This is the equation of a circle of radius $\frac{1}{\sqrt{2}|k|}$ centered about the point $\left(\frac{1}{2k}, \frac{1}{2k}\right)$. Noting that the distance of the point $\left(\frac{1}{2k}, \frac{1}{2k}\right)$ from the origin is precisely $\frac{1}{\sqrt{2k^2}}$, we can conclude that equipotential surfaces are circles that always contain the origin (0,0), and whose their centers will lie along the line $x = y$.



The flow lines of the gradient field $\mathbf{F} = -\nabla V$ will always be anti-parallel to ∇V which will always be perpendicular to the equipotential surfaces (this we know from Section 2.5). Thus, to sketch the vector field $\mathbf{F}$ we can sketch the equipotential surfaces and then draw vectors that are perpendicular to the equipotential surfaces.



∎

4.3.5. Show that $\sigma(t) = \left(e^{2t}, \ln|t|, 1/t\right)$ for $t \neq 0$ is a flow line of the velocity vector field $\mathbf{F}(x,y,z) = \left(2x, z, -z^2\right)$.

- Well,

$$\begin{aligned} \frac{d\sigma_x}{dt}(t) &= 2e^{2t} = 2\sigma_x(t) = F_x(\sigma(t)) \\ \frac{d\sigma_y}{dt}(t) &= \frac{1}{t} = \sigma_z(t) = F_y(\sigma(t)) \\ \frac{d\sigma_z}{dt} &= -\frac{1}{t^2} = -(\sigma_z(t))^2 = F_z(\sigma(t)) \end{aligned}$$

Thus

$$\frac{d\sigma}{dt}(t) = \mathbf{F}(\sigma(t))$$

and so $\sigma(t)$ is a flow line of $\mathbf{F}$. ∎

Section 4.4

4.4.1. Compute the curl, $\nabla \times \mathbf{F}$, of each of the following vector fields.

(a) $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

- We have

$$\begin{aligned} \nabla \times \mathbf{F} &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \\ &= (0-0, 0-0, 0-0) \\ &= (0,0,0) \end{aligned}$$

∎

(b) $\mathbf{F}(x,y,z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

- We have

$$\begin{aligned}\nabla \times \mathbf{F} &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \\ &= (x - x, y - y, z - z) \\ &= (0, 0, 0)\end{aligned}$$

∎

(c) $\mathbf{F}(x, y, z) = \left(x^2 + y^2 + z^2\right)(3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k})$

-

$$\begin{aligned}\nabla \times \mathbf{F} &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \\ &= (10y - 8z, 6z - 10x, 8x - 6y)\end{aligned}$$

∎

4.4.2. Compute the divergence of each of the vector fields in Exercise 1.

(a)

-

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \nabla \cdot (x, y, z) \\ &= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) \\ &= 1 + 1 + 1 \\ &= 3\end{aligned}$$

∎

(b)

-

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \nabla \cdot (yz, xz, xy) \\ &= \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) \\ &= 0 + 0 + 0 \\ &= 0\end{aligned}$$

∎

(c)

-

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \nabla \cdot \left(3x^2 + 3y^2 + 3z^2, 4x^2 + 4y^2 + 4z^2, 5x^2 + 5y^2 + 5z^2\right) \\ &= \frac{\partial}{\partial x}\left(3x^2 + 3y^2 + 3z^2\right) + \frac{\partial}{\partial y}\left(4x^2 + 4y^2 + 4z^2\right) + \frac{\partial}{\partial z}\left(5x^2 + 5y^2 + 5z^2\right) \\ &= 6x + 8y + 10z\end{aligned}$$

∎

4.4.3. Let $\mathbf{F}(x, y, z) = 3x^2y\mathbf{i} + \left(x^3 + y^3\right)\mathbf{j}$.

(a) Verify that $\nabla \times \mathbf{F} = 0$.

- $$\begin{aligned}\nabla \times \mathbf{F} &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \\ &= \left(0-0, 0-0, 3x^2 - 3x^2\right) \\ &= (0,0,0)\end{aligned}$$

  ∎

(b) Find a function f such that $\mathbf{F} = \nabla f$.

- We need to find a function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ such that
$$\begin{aligned}\frac{\partial f}{\partial x} &= 3x^2 y \\ \frac{\partial f}{\partial y} &= x^3 + y^3 \\ \frac{\partial f}{\partial z} &= 0\end{aligned}$$
Now the most general function f of x, y, z satisfying the first equation in (B1) will be of the form
$$f(x,y,z) = \int 3x^2 y \, dx + h_1(y,z) = x^3 y + h_1(y,z) \qquad (B2)$$
Here $h_1(y,z)$ is an arbitrary function of y and z.
The most general function satisfying the second equation in (B2) will be of the form
$$f(x,y,z) = \int \left(x^3 + y^3\right) dy + h_2(x,z) = x^3 y + \frac{1}{4} y^4 + h_2(x,z) \qquad B3$$
where $h_2(x,z)$ is an arbitrary function of x and z.
The most general function satisfying the third equation (B3) will be of the form
$$f(x,y,z) = \int 0 \cdot dz + h_3(x,y) = h_3(x,y) \quad . \qquad (B4)$$
Now the function f that we seek must satisfy (B2), (B3), and (B4) simultaneously. Equation (B2) tells us that the x dependence of f lies solely in a term of the form $x^3 y$; equation (B3) tells us that the y dependence of f lies solely in the sum of two terms $x^3 y + \frac{1}{4} y^4$; and equation (B4) tells us that f does not depend at all on z. We can thus conclude that any function of the form
$$f(x,y,z) = x^3 y + \frac{1}{4} y^4 + C$$
will be a solution of $\nabla f = \mathbf{F}$. ∎

(c) Is it true that for a vector field $\mathbf{F}$ such a function can exist only if $\nabla \times \mathbf{F} = 0$?

- Suppose $\mathbf{F} = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$. Then
$$\nabla \times \mathbf{F} = \left(\frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y}, \frac{\partial}{\partial z}\frac{\partial f}{\partial x} - \frac{\partial}{\partial x}\frac{\partial f}{\partial z}, \frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\right)$$
Now by Theorem 15 (Section 2.6), if f is of class C^2, then
$$0 = \frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y} = \frac{\partial}{\partial z}\frac{\partial f}{\partial x} - \frac{\partial}{\partial x}\frac{\partial f}{\partial z} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x}.$$
We conclude that if $\nabla \times \mathbf{F} \neq 0$, there can be no function of class C^2 such that $\mathbf{F} = \nabla f$. ∎

4.4.4. Show that $\mathbf{F} = y\,(\cos(x))\,\mathbf{i} + x\,(\sin(y))\,\mathbf{j}$ is *not* a gradient field.

- Suppose that $\mathbf{F} = \nabla f$. Then

$$\begin{aligned} \frac{\partial f}{\partial x} &= y\cos(x) \\ \frac{\partial f}{\partial y} &= x\sin(y) \end{aligned}$$

Each of the two functions on the right hand side are perfectly continuous, and moreover, their partial derivatives exist and are continuos for all x and y. Therefore, f is at least of class C^2. But then, by Theorem 15 of Section 2.6, we must have

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} \quad .$$

But

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \cos(x) \neq \sin(x) = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} \quad .$$

We conclude that $\mathbf{F}$ can not be a gradient field. ∎

Section 4.5

4.5.1. Suppose $\nabla \cdot \mathbf{F} = 0$ and $\nabla \cdot \mathbf{G} = 0$. Which of the following vector fields necessarily have zero divergence?

(a) $\mathbf{F} + \mathbf{G}$

- By Identity 5 on page 283 we have

$$\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G} = 0 + 0 = 0 \quad .$$

∎

(b) $\mathbf{F} \times \mathbf{G}$

- By Identity 9 on page 283 we have

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}) \quad .$$

The expression of the right hand side does not necessarily vanish (even if $0 = \nabla \cdot \mathbf{F} = \nabla \cdot \mathbf{G}$). For example, if

$$\begin{aligned} \mathbf{F} &= (-y, x, 0) \\ \mathbf{G} &= (0, 0, 1) \end{aligned}$$

Then

$$0 = \nabla \cdot \mathbf{F} = \nabla \cdot \mathbf{G}$$

and

$$\begin{aligned} \nabla \cdot (\mathbf{F} \times \mathbf{G}) &= \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}) \\ &= (0,0,1) \cdot (0,0,2) - (-y, x, 0) \cdot (0,0,0) \\ &= 2 \end{aligned}$$

∎

(c) $(\mathbf{F} \cdot \mathbf{G})\,\mathbf{F}$

- By Identities 8 and 7 on page 283 we have

$$
\begin{aligned}
\nabla\cdot((\mathbf{F}\cdot\mathbf{G})\,\mathbf{F}) &= (\mathbf{F}\cdot\mathbf{G})\,(\nabla\cdot\mathbf{F})+\mathbf{F}\cdot\nabla\,(\mathbf{F}\cdot\mathbf{G})\\
&= (\mathbf{F}\cdot\mathbf{G})\,(\nabla\cdot\mathbf{F})\\
&\quad +\mathbf{F}\cdot((\mathbf{F}\cdot\nabla)\,\mathbf{G}+(\mathbf{G}\cdot\nabla)\,\mathbf{F}+\mathbf{F}\times(\nabla\times\mathbf{G})+\mathbf{G}\times(\nabla\times\mathbf{F}))\\
&= 0+\mathbf{F}\cdot((\mathbf{F}\cdot\nabla)\,\mathbf{G}+(\mathbf{G}\cdot\nabla)\,\mathbf{F}+\mathbf{F}\times(\nabla\times\mathbf{G})+\mathbf{G}\times(\nabla\times\mathbf{F}))
\end{aligned}
$$

The expression of the right hand side does not necessarily vanish (even if $0=\nabla\cdot\mathbf{F}=\nabla\cdot\mathbf{G}$). ∎

4.5.2. Prove the following identities.

(a) $\nabla\,(\mathbf{F}\cdot\mathbf{G})=(\mathbf{F}\cdot\nabla)\,\mathbf{G}+(\mathbf{G}\cdot\nabla)\,\mathbf{F}+\mathbf{F}\times(\nabla\times\mathbf{G})+\mathbf{G}\times(\nabla\times\mathbf{F})$

- By virtue of the product rule the left hand side is equivalent to

$$
\begin{aligned}
LHS &= \nabla\,(\mathbf{F}\cdot\mathbf{G})=\\
&= \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)(F_xG_x+F_yG_y+F_zG_z)\\
&= \left(\frac{\partial F_x}{\partial x}G_x+F_x\frac{\partial G_x}{\partial x}+\frac{\partial F_y}{\partial x}G_y+F_y\frac{\partial G_y}{\partial x}+\frac{\partial F_z}{\partial x}G_z+F_z\frac{\partial G_z}{\partial x}\right)\mathbf{i}\\
&\quad +\left(\frac{\partial F_x}{\partial y}G_x+F_x\frac{\partial G_x}{\partial y}+\frac{\partial F_y}{\partial y}G_y+F_y\frac{\partial G_y}{\partial y}+\frac{\partial F_z}{\partial y}G_z+F_z\frac{\partial G_z}{\partial y}\right)\mathbf{j}\\
&\quad +\left(\frac{\partial F_x}{\partial z}G_x+F_x\frac{\partial G_x}{\partial z}+\frac{\partial F_y}{\partial z}G_y+F_y\frac{\partial G_y}{\partial z}+\frac{\partial F_z}{\partial z}G_z+F_z\frac{\partial G_z}{\partial z}\right)\mathbf{k}
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
(\mathbf{F}\cdot\nabla)\,\mathbf{G} &= \left(F_x\frac{\partial}{\partial x}+F_y\frac{\partial}{\partial y}+F_z\frac{\partial}{\partial z}\right)(G_x,G_y,G_z)\\
&= \left(F_x\frac{\partial G_x}{\partial x}+F_y\frac{\partial G_x}{\partial y}+F_z\frac{\partial G_x}{\partial z}\right)\mathbf{i}\\
&\quad +\left(F_x\frac{\partial G_y}{\partial x}+F_y\frac{\partial G_y}{\partial y}+F_z\frac{\partial G_y}{\partial z}\right)\mathbf{j}\\
&\quad +\left(F_x\frac{\partial G_z}{\partial x}+F_y\frac{\partial G_z}{\partial y}+F_z\frac{\partial G_z}{\partial z}\right)\mathbf{k}
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{G}\cdot\nabla)\,\mathbf{F} &= \left(G_x\frac{\partial}{\partial x}+G_y\frac{\partial}{\partial y}+G_z\frac{\partial}{\partial z}\right)(F_x,F_y,F_z)\\
&= \left(G_x\frac{\partial F_x}{\partial x}+G_y\frac{\partial F_x}{\partial y}+G_z\frac{\partial F_x}{\partial z}\right)\mathbf{i}\\
&\quad +\left(G_x\frac{\partial F_y}{\partial x}+G_y\frac{\partial F_y}{\partial y}+G_z\frac{\partial F_y}{\partial z}\right)\mathbf{j}\\
&\quad +\left(G_x\frac{\partial F_z}{\partial x}+G_y\frac{\partial F_z}{\partial y}+G_z\frac{\partial F_z}{\partial z}\right)\mathbf{k}
\end{aligned}
$$

$$\begin{aligned}
\mathbf{F}\times(\nabla\times\mathbf{G}) &= (F_x, F_y, F_z)\times\left(\frac{\partial G_z}{\partial y}-\frac{\partial G_y}{\partial z}, \frac{\partial G_x}{\partial z}-\frac{\partial G_z}{\partial x}, \frac{\partial G_y}{\partial x}-\frac{\partial G_x}{\partial y}\right)\\
&= \left(F_y\frac{\partial G_y}{\partial x}-F_y\frac{\partial G_x}{\partial y}-F_z\frac{\partial G_x}{\partial z}+F_z\frac{\partial Gz}{\partial x}\right)\mathbf{i}\\
&\quad+\left(F_z\frac{\partial G_z}{\partial y}-F_z\frac{\partial G_y}{\partial z}-F_x\frac{\partial G_y}{\partial x}+F_x\frac{\partial G_x}{\partial y}\right)\mathbf{j}\\
&\quad+\left(F_x\frac{\partial G_x}{\partial z}-F_x\frac{\partial G_z}{\partial x}-F_y\frac{\partial G_z}{\partial y}+F_y\frac{\partial G_y}{\partial z}\right)\mathbf{k}
\end{aligned}$$

$$\begin{aligned}
\mathbf{G}\times(\nabla\times\mathbf{F}) &= (G_x, G_y, G_z)\times\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\\
&= \left(G_y\frac{\partial F_y}{\partial x}-G_y\frac{\partial F_x}{\partial y}-G_z\frac{\partial F_x}{\partial z}+G_z\frac{\partial Fz}{\partial x}\right)\mathbf{i}\\
&\quad+\left(G_z\frac{\partial F_z}{\partial y}-G_z\frac{\partial F_y}{\partial z}-G_x\frac{\partial F_y}{\partial x}+G_x\frac{\partial F_x}{\partial y}\right)\mathbf{j}\\
&\quad+\left(G_x\frac{\partial F_x}{\partial z}-G_x\frac{\partial F_z}{\partial x}-G_y\frac{\partial F_z}{\partial y}+G_y\frac{\partial F_y}{\partial z}\right)\mathbf{k}
\end{aligned}$$

And so the right hand side of (a) is

$$\begin{aligned}
RHS &= (\mathbf{F}\cdot\nabla)\mathbf{G}+(\mathbf{G}\cdot\nabla)\mathbf{F}+\mathbf{F}\times(\nabla\times\mathbf{G})+\mathbf{G}\times(\nabla\times\mathbf{F})\\
&= \Bigg(F_x\frac{\partial G_x}{\partial x}+F_y\frac{\partial G_x}{\partial y}+F_z\frac{\partial G_x}{\partial z}+G_x\frac{\partial F_x}{\partial x}+G_y\frac{\partial F_x}{\partial y}+G_z\frac{\partial F_x}{\partial z}+F_y\frac{\partial G_y}{\partial x}\\
&\quad -F_y\frac{\partial G_x}{\partial y}-F_z\frac{\partial G_x}{\partial z}+F_z\frac{\partial Gz}{\partial x}+G_y\frac{\partial F_y}{\partial x}-G_y\frac{\partial F_x}{\partial y}-G_z\frac{\partial F_x}{\partial z}+G_z\frac{\partial Fz}{\partial x}\Bigg)\mathbf{i}\\
&\quad+\Bigg(F_x\frac{\partial G_y}{\partial x}+F_y\frac{\partial G_y}{\partial y}+F_z\frac{\partial G_y}{\partial z}+G_x\frac{\partial F_y}{\partial x}+G_y\frac{\partial F_y}{\partial y}+G_z\frac{\partial F_y}{\partial z}+F_z\frac{\partial G_z}{\partial y}\\
&\quad -F_z\frac{\partial G_y}{\partial z}-F_x\frac{\partial G_y}{\partial x}+F_x\frac{\partial G_x}{\partial y}+G_z\frac{\partial F_z}{\partial y}-G_z\frac{\partial F_y}{\partial z}-G_x\frac{\partial F_y}{\partial x}+G_x\frac{\partial F_x}{\partial y}\Bigg)\mathbf{j}\\
&\quad\Bigg(F_x\frac{\partial G_z}{\partial x}+F_y\frac{\partial G_z}{\partial y}+F_z\frac{\partial G_z}{\partial z}+G_x\frac{\partial F_z}{\partial x}+G_y\frac{\partial F_z}{\partial y}+G_z\frac{\partial F_z}{\partial z}+F_x\frac{\partial G_x}{\partial z}\\
&\quad -F_x\frac{\partial G_z}{\partial x}-F_y\frac{\partial G_z}{\partial y}+F_y\frac{\partial G_y}{\partial z}+G_x\frac{\partial F_x}{\partial z}-G_x\frac{\partial F_z}{\partial x}-G_y\frac{\partial F_z}{\partial y}+G_y\frac{\partial F_y}{\partial z}\Bigg)\mathbf{k}\\
&= \left(\frac{\partial F_x}{\partial x}G_x+F_x\frac{\partial G_x}{\partial x}+\frac{\partial F_y}{\partial x}G_y+F_y\frac{\partial G_y}{\partial x}+\frac{\partial F_z}{\partial x}G_z+F_z\frac{\partial G_z}{\partial x}\right)\mathbf{i}\\
&\quad+\left(\frac{\partial F_x}{\partial y}G_x+F_x\frac{\partial G_x}{\partial y}+\frac{\partial F_y}{\partial y}G_y+F_y\frac{\partial G_y}{\partial y}+\frac{\partial F_z}{\partial y}G_z+F_z\frac{\partial G_z}{\partial y}\right)\mathbf{j}\\
&\quad+\left(\frac{\partial F_x}{\partial z}G_x+F_x\frac{\partial G_x}{\partial z}+\frac{\partial F_y}{\partial z}G_y+F_y\frac{\partial G_y}{\partial z}+\frac{\partial F_z}{\partial z}G_z+F_z\frac{\partial G_z}{\partial z}\right)\mathbf{k}
\end{aligned}$$

which is equivalent to the left hand side of identity (a). ∎

(b) $\nabla\cdot(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot(\nabla\times\mathbf{F}) - \mathbf{F}\cdot(\nabla\times\mathbf{G})$

- We have

$$
\begin{aligned}
\nabla \cdot (\mathbf{F} \times \mathbf{G}) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (F_y G_z - F_z G_y, F_z G_x - F_x G_z, F_x G_y - F_y G_x) \\
&= G_z \frac{\partial F_y}{\partial x} + F_y \frac{\partial G_z}{\partial x} - G_y \frac{\partial F_z}{\partial x} - F_z \frac{\partial G_y}{\partial x} \\
&\quad + G_x \frac{\partial F_z}{\partial y} + F_z \frac{\partial G_x}{\partial y} - G_z \frac{\partial F_x}{\partial y} - F_x \frac{\partial G_z}{\partial y} \\
&\quad + G_y \frac{\partial F_x}{\partial z} + F_x \frac{\partial G_y}{\partial z} - G_x \frac{\partial F_y}{\partial z} - F_y \frac{\partial G_x}{\partial z}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{G} \cdot (\nabla \times \mathbf{F}) &= (G_x, G_y, G_z) \times \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \\
&= G_x \frac{\partial F_z}{\partial y} - G_x \frac{\partial F_y}{\partial z} + G_y \frac{\partial F_x}{\partial z} - G_y \frac{\partial F_z}{\partial x} + G_z \frac{\partial F_y}{\partial x} - G_z \frac{\partial F_x}{\partial y}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{F} \cdot (\nabla \times \mathbf{G}) &= (F_x, F_y, F_z) \times \left(\frac{\partial G_z}{\partial y} - \frac{\partial G_y}{\partial z}, \frac{\partial G_x}{\partial z} - \frac{\partial G_z}{\partial x}, \frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y}\right) \\
&= F_x \frac{\partial G_z}{\partial y} - F_x \frac{\partial G_y}{\partial z} + F_y \frac{\partial G_x}{\partial z} - F_y \frac{\partial G_z}{\partial x} + F_z \frac{\partial G_y}{\partial x} - F_z \frac{\partial G_x}{\partial y}
\end{aligned}
$$

So

$$
\begin{aligned}
\mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}) &= G_x \frac{\partial F_z}{\partial y} - G_x \frac{\partial F_y}{\partial z} + G_y \frac{\partial F_x}{\partial z} - G_y \frac{\partial F_z}{\partial x} + G_z \frac{\partial F_y}{\partial x} - G_z \frac{\partial F_x}{\partial y} \\
&\quad - F_x \frac{\partial G_z}{\partial y} + F_x \frac{\partial G_y}{\partial z} - F_y \frac{\partial G_x}{\partial z} + F_y \frac{\partial G_z}{\partial x} - F_z \frac{\partial G_y}{\partial x} + F_z \frac{\partial G_x}{\partial y} \\
&= G_z \frac{\partial F_y}{\partial x} + F_y \frac{\partial G_z}{\partial x} - G_y \frac{\partial F_z}{\partial x} - F_z \frac{\partial G_y}{\partial x} \\
&\quad + G_x \frac{\partial F_z}{\partial y} + F_z \frac{\partial G_x}{\partial y} - G_z \frac{\partial F_x}{\partial y} - F_x \frac{\partial G_z}{\partial y} \\
&\quad + G_y \frac{\partial F_x}{\partial z} + F_x \frac{\partial G_y}{\partial z} - G_x \frac{\partial F_y}{\partial z} - F_y \frac{\partial G_x}{\partial z} \\
&= \nabla \cdot (\mathbf{F} \cdot \mathbf{G})
\end{aligned}
$$

∎

(c) $\nabla \times (f\mathbf{F}) = f(\nabla \times \mathbf{F}) + \nabla f \times \mathbf{F}$

- We have

$$
\begin{aligned}
\nabla \times (fF_x, fF_y, fF_z) &= \left(F_z\frac{\partial f}{\partial y} + f\frac{\partial F_z}{\partial y} - F_y\frac{\partial f}{\partial z} - f\frac{\partial F_y}{\partial z}\right)\mathbf{i} \\
&\quad + \left(F_x\frac{\partial f}{\partial z} + f\frac{\partial F_x}{\partial z} - F_z\frac{\partial f}{\partial x} - f\frac{\partial F_z}{\partial x}\right)\mathbf{j} \\
&\quad + \left(F_y\frac{\partial f}{\partial x} + f\frac{\partial F_y}{\partial x} - F_x\frac{\partial f}{\partial y} - f\frac{\partial F_x}{\partial y}\right)\mathbf{k} \\
&= f\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \\
&\quad + \left((\nabla f)_y F_z - (\nabla f)_z F_y\right)\mathbf{i} \\
&\quad + ((\nabla f)_z F_x - (\nabla f)_x F_z)\mathbf{j} \\
&\quad + \left((\nabla f)_x F_y - (\nabla f)_y F_x\right) \\
&= f(\nabla \times \mathbf{F}) + \nabla f \times \mathbf{F}
\end{aligned}
$$

∎

4.5.3. Let $\mathbf{F} = \left(2xz^2, 1, y^3zx\right)$, $\mathbf{G} = \left(x^2, y^2, z^2\right)$, and $f = x^2y$. Compute the following quantities.

(a) ∇f

-

$$\nabla f = \left(2xy, x^2, 0\right)$$

∎

(b) $\nabla \times \mathbf{F}$

-

$$\nabla \times \mathbf{F} = \left(3y^2zx, 4xz - y^3z, 0\right)$$

∎

(c) $(\mathbf{F} \cdot \nabla)\,\mathbf{G}$

-

$$
\begin{aligned}
(\mathbf{F} \cdot \nabla)\,\mathbf{G} &= \left(2xz^2\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + y^3zx\frac{\partial}{\partial z}\right) \cdot \left(x^2, y^2, z^2\right) \\
&= \left(4x^2z^2, 2y, 2y^3z^2x\right) \quad .
\end{aligned}
$$

∎

(d) $\mathbf{F} \cdot (\nabla f)$

-

$$
\begin{aligned}
\mathbf{F} \cdot (\nabla f) &= \left(2xz^2, 1, y^3zx\right) \cdot \left(2xy, x^2, 0\right) \\
&= 4x^2yz^2 + x^2
\end{aligned}
$$

∎

(e) $\mathbf{F} \times \nabla f$

-

$$
\begin{aligned}
\mathbf{F} \times (\nabla f) &= \left(2xz^2, 1, y^3zx\right) \times \left(2xy, x^2, 0\right) \\
&= \left(-y^3zx^3, 2y^4x^2z, 2x^3z^2 - 2xy\right)
\end{aligned}
$$

∎

4.5.4. Let $\mathbf{F}$ be a general vector field. Does $\nabla \times \mathbf{F}$ have to be perpendicular to $\mathbf{F}$.

- No, consider the vector field

$$\mathbf{F}(x, y, z) = (-y, x, 1) \quad .$$

We have

$$\nabla \times \mathbf{F} = (0 - 0, 0 - 0, 1 - (-1)) = (0, 0, 2)$$

So,

$$\mathbf{F} \cdot (\nabla \times \mathbf{F}) = (-y, x, 1) \cdot (0, 0, 2) = 2 \neq 0 \quad .$$

Thus, $\nabla \times \mathbf{F}$ is not perpendicular to $\mathbf{F}$. ∎