Math 4013
Solutions to Homework Problems from Chapter 3

Section 3.1

3.1.1. Compute the second partial derivatives $\partial^2 f/\partial x^2$, $\partial^2 f/\partial x\partial y$, $\partial^2 f/\partial y\partial x$, $\partial^2 f/\partial y^2$ for each of the following functions. Verify Theorem 15 in each case.

(a) $f(x,y) = 2xy/\left(x^2+y^2\right)^2$, $(x,y) \neq 0$.

•

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{2y\left(x^2+y^2\right)^2 - (2xy)2\left(x^2+y^2\right)(2x)}{\left(x^2+y^2\right)^4} \\
&= \frac{2y^3 - 6x^2y}{\left(x^2+y^2\right)^3}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{2x\left(x^2+y^2\right)^2 - (2xy)2\left(x^2+y^2\right)(2y)}{\left(x^2+y^2\right)^4} \\
&= \frac{2x^3 - 6xy^2}{\left(x^2+y^2\right)^3}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial y}\frac{\partial f}{\partial x} &= \frac{\left(6y^2-6x^2\right)\left(x^2+y^2\right)^3 - \left(2y^3-6x^2y\right)3\left(x^2+y^2\right)^2(2y)}{\left(x^2+y^2\right)^6} \\
&= \frac{-6x^4-6y^4+36x^2y^2}{\left(x^2+y^2\right)^4}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{\partial f}{\partial y} &= \frac{\left(6x^2-6y^2\right)\left(x^2+y^2\right)^3 - \left(2x^3-6xy^2\right)3\left(x^2+y^2\right)^2(2y)}{\left(x^2+y^2\right)^6} \\
&= \frac{-6x^4-6y^4+36x^2y^2}{\left(x^2+y^2\right)^4}
\end{aligned}$$

Note that

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} \quad .$$

$\blacksquare$

(b) $f(x,y,z) = e^z + (1/x) + xe^{-y}$, $x \neq 0$.

•

$$\begin{aligned}
\frac{\partial f}{\partial x} &= -\frac{1}{x^2} + e^{-y} \\
\frac{\partial}{\partial y}\frac{\partial f}{\partial x} &= -e^{-y} \\
\frac{\partial}{\partial z}\frac{\partial f}{\partial x} &= 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial y} &= -xe^{-y} \\
\frac{\partial}{\partial x}\frac{\partial f}{\partial y} &= -e^{-y} \\
\frac{\partial}{\partial z}\frac{\partial f}{\partial y} &= 0
\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial z} &= e^z \\ \frac{\partial}{\partial x}\frac{\partial f}{\partial z} &= 0 \\ \frac{\partial}{\partial y}\frac{\partial f}{\partial z} &= 0\end{aligned}$$

Note that

$$\begin{aligned}\frac{\partial}{\partial x}\frac{\partial f}{\partial y} &= -e^{-y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} \quad , \\ \frac{\partial}{\partial x}\frac{\partial f}{\partial z} &= 0 = \frac{\partial}{\partial z}\frac{\partial f}{\partial x} \\ \frac{\partial}{\partial y}\frac{\partial f}{\partial z} &= 0 = \frac{\partial}{\partial z}\frac{\partial f}{\partial y}\end{aligned}$$

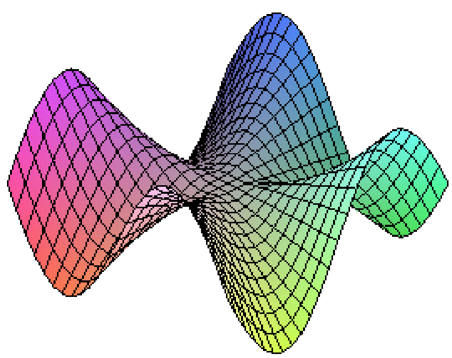
∎

3.1.2. Let

$$f(x,y) = \begin{cases} xy\left(x^2-y^2\right)/\left(x^2+y^2\right) & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = 0 \end{cases}$$

(a) If $(x,y) \neq 0$, calculate $\partial f/\partial x$ and $\partial f/\partial y$.

- This function, despite the apparent problem at origin, is actually perfectly continuous. Its graph looks like



For all points $(x,y) \neq 0$, f is differentiable since there it is a ratio of two polynomials for which the denominator never vanishes. We can thus calculate the partial derivatives of f at points different than $(0,0)$ using the quorient rule.

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\left(3x^2y-y^3\right)\left(x^2+y^2\right)-\left(x^3y-xy^3\right)(2x)}{\left(x^2+y^2\right)^2} \\ &= \frac{x^4y+4x^2y^3-y^5}{\left(x^2+y^2\right)^2} \\ \frac{\partial f}{\partial y} &= \frac{\left(x^3-3xy^2\right)\left(x^2+y^2\right)-\left(x^3y-xy^3\right)(2y)}{\left(x^2+y^2\right)^2} \\ &= \frac{x^5-4x^3y^2-xy^4}{\left(x^2+y^2\right)^2}\end{aligned}$$

∎

(b) Show that

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = 0 = \left.\frac{\partial f}{\partial y}\right|_{(0,0)}$$

- At the point (0,0) we will have to be more careful, because $f(x,y)$ is not obviously differentiable there. Applying the definitions of the partial derivatives there we compute

$$\begin{aligned}
\left.\frac{\partial f}{\partial x}\right|_{(0,0)} &= \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} \\
&= \lim_{h\to 0} \frac{0-0}{h} \\
&= \lim_{h\to 0} 0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\left.\frac{\partial f}{\partial y}\right|_{(0,0)} &= \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} \\
&= \lim_{h\to 0} \frac{0-0}{h} \\
&= \lim_{h\to 0} 0 \\
&= 0
\end{aligned}$$

∎

(c) Show that

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(0,0)} = 1 \quad , \quad \left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(0,0)} = -1 \quad .$$

- Again, we have to be a little careful evaluating the partial derivatives at the point (0,0). We have

$$\begin{aligned}
\left.\frac{\partial}{\partial y}\frac{\partial f}{\partial x}\right|_{(0,0)} &= \lim_{h\to 0} \frac{\left.\frac{\partial f}{\partial x}\right|_{(0,h)} - \left.\frac{\partial f}{\partial x}\right|_{(0,0)}}{h} \\
&= \lim_{h\to 0} \frac{-h-0}{h} \\
&= -1
\end{aligned}$$

$$\begin{aligned}
\left.\frac{\partial}{\partial x}\frac{\partial f}{\partial y}\right|_{(0,0)} &= \lim_{h\to 0} \frac{\left.\frac{\partial f}{\partial y}\right|_{(h,0)} - \left.\frac{\partial f}{\partial y}\right|_{(0,0)}}{h} \\
&= \lim_{h\to 0} \frac{h-0}{h} \\
&= 1
\end{aligned}$$

(d) What went wrong? Why are the mixed partials not equal?

The second partial derivatives exist, but they are not continuous as functions of two variables. Therefore, Theorem 15 can not be applied in this case. ∎

3.1.3. A function $u = f(x,y)$ with continuous second partial derivatives satisfying Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is called a *harmonic function*. Show that $u(x,y) = x^3 - 3xy^2$ is harmonic.

- $$\begin{aligned}\frac{\partial u}{\partial x} &= 3x^2 - 3y^2 \\ \frac{\partial^2 u}{\partial x^2} &= 6x \\ \frac{\partial u}{\partial y} &= -6xy \\ \frac{\partial^2 u}{\partial y^2} &= -6x\end{aligned}$$

  Therefore
  $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0 \quad ,$$
  and so u is harmonic. ∎

Section 3.2

We shall use the following formula for the second order Taylor series of a function $f : \mathbb{R}^n \to \mathbb{R}$ about a point $\mathbf{a} \in \mathbb{R}^n$;

$$f(\mathbf{x}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + \frac{1}{2} (\mathbf{x} - \mathbf{a})^T \mathbf{H} f(\mathbf{a}) (\mathbf{x} - \mathbf{a}) + \left(\|\mathbf{x} - \mathbf{a}\|^3\right)$$

where

$$\begin{aligned}\mathbf{x} - \mathbf{a} &= \begin{pmatrix} x_1 - a_1 \\ x_2 - a_2 \\ \vdots \\ x_n - a_n \end{pmatrix} \\ (\mathbf{x} - \mathbf{a})^T &= (x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n) \\ \nabla f(\mathbf{a}) &= \left(\frac{\partial f}{\partial x_1}(\mathbf{a}), \frac{\partial f}{\partial x_2}(\mathbf{a}), \cdots, \frac{\partial f}{\partial x_n}(\mathbf{a}) \right)\end{aligned}$$

and $\mathbf{H} f(\mathbf{a})$ is the $n \times n$ matrix

$$\mathbf{H} f(\mathbf{a}) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(\mathbf{a}) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(\mathbf{a}) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(\mathbf{a}) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(\mathbf{a}) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(\mathbf{a}) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(\mathbf{a}) \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(\mathbf{a}) & \frac{\partial^2 f}{\partial x_n \partial x_2}(\mathbf{a}) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n}(\mathbf{a}) \end{pmatrix}$$

3.2.1. Determine the second order Taylor formula for $f(x, y) = (x + y)^2$ about $(0, 0)$..

- We have

$$\begin{aligned} f(\mathbf{a}) &= (0 + 0)^2 = 0 \\ \mathbf{x} - \mathbf{a} &= \begin{pmatrix} x - 0 \\ y - 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \\ (\mathbf{x} - \mathbf{a})^T &= (x - 0, y - 0) = (x \quad y) \\ \nabla f(\mathbf{a}) &= (2x + 2y, 2x + 2y) \big|_{(0,0)} = (0, 0) \\ \mathbf{H} f(\mathbf{a}) &= \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(0,0) & \frac{\partial^2 f}{\partial x \partial y}(0,0) \\ \frac{\partial^2 f}{\partial y \partial x}(0,0) & \frac{\partial^2 f}{\partial y^2}(0,0) \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \Bigg|_{(0,0)} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \end{aligned}$$

and so

$$\begin{aligned}
f(\mathbf{x}) &= 0 + (0,0)\cdot(x,y) + \frac{1}{2}\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \mathcal{O}\left(\|\mathbf{x}-\mathbf{a}\|^3\right) \\
&= 0 + 0 + \frac{1}{2}\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 2x+2y \\ 2x+2y \end{pmatrix} + \mathcal{O}\left(\|\mathbf{x}\|^3\right) \\
&= \frac{1}{2}\left(2x^2 + 2xy + 2yx + 2y^2\right) + \mathcal{O}\left(\|\mathbf{x}\|^3\right) \\
&= x^2 + 2xy + y^2 + \mathcal{O}\left(\|\mathbf{x}\|^3\right)
\end{aligned}$$

∎

3.2.2 Determine the second order Taylor formula for $f(x,y) = 1/\left(x^2+y^2+1\right)$ about $(0,0)$.

- We have

$$\begin{aligned} f(\mathbf{a}) &= \frac{1}{1} = 1 \\ \frac{\partial f}{\partial x}(\mathbf{a}) &= \left.\frac{-2x}{\left(x^2+y^2+1\right)^2}\right|_{(0,0)} = 0 \\ \frac{\partial f}{\partial y}(\mathbf{a}) &= \left.\frac{-2y}{\left(x^2+y^2+1\right)^2}\right|_{(0,0)} = 0 \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2}(\mathbf{a}) &= \left.\left[\frac{-2}{\left(x^2+y^2+1\right)^2} + \frac{8x^2}{\left(x^2+y^2+1\right)^4}\right]\right|_{(0,0)} = -2 \\ \frac{\partial^2 f}{\partial x \partial y}(\mathbf{a}) &= \left.\left[\frac{4xy}{\left(x^2+y^2+1\right)^4}\right]\right|_{(0,0)} = 0 \\ \frac{\partial^2 f}{\partial y \partial x}(\mathbf{a}) &= \left.\left[\frac{4xy}{\left(x^2+y^2+1\right)^4}\right]\right|_{(0,0)} = 0 \\ \frac{\partial^2 f}{\partial y^2}(\mathbf{a}) &= \left.\left[\frac{-2}{\left(x^2+y^2+1\right)^2} + \frac{8y^2}{\left(x^2+y^2+1\right)^4}\right]\right|_{(0,0)} = -2 \end{aligned}$$

So

$$\begin{aligned} f(\mathbf{x}) &= f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x}-\mathbf{a}) + \frac{1}{2}(\mathbf{x}-\mathbf{a})^T \mathbf{H} f(\mathbf{a})(\mathbf{x}-\mathbf{a}) + \mathcal{O}\left(\|\mathbf{x}-\mathbf{a}\|^3\right) \\ &= 1 + (0,0)\cdot(x,y) + \frac{1}{2}(x \quad y)\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \mathcal{O}\left(\|\mathbf{x}\|^3\right) \\ &= 1 + \frac{1}{2}(x \quad y)\begin{pmatrix} -2x \\ -2y \end{pmatrix} + \mathcal{O}\left(\|\mathbf{x}\|^3\right) \\ &= 1 + \frac{1}{2}\left(-2x^2 - 2y^2\right) + \mathcal{O}\left(\|\mathbf{x}\|^3\right) \\ &= 1 - x^2 - y^2 + \mathcal{O}\left(\|\mathbf{x}\|^3\right) \end{aligned}$$

∎

3.2.3. Determine the second order Taylor formula for $f(x,y) = e^{x+y}$ about $(0,0)$.

- We have

$$\begin{aligned} f(\mathbf{a}) &= e^{0+0} = 1 \\ \frac{\partial f}{\partial x}(\mathbf{a}) &= \left. e^{x+y}\right|_{(0,0)} = 1 \\ \frac{\partial f}{\partial y}(\mathbf{a}) &= \left. e^{x+y}\right|_{(0,0)} = 1 \end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x^2}(\mathbf{a}) &= \left. e^{x+y}\right|_{(0,0)} = 1 \\
\frac{\partial^2 f}{\partial x \partial y}(\mathbf{a}) &= \left. e^{x+y}\right|_{(0,0)} = 1 \\
\frac{\partial^2 f}{\partial y \partial x}(\mathbf{a}) &= \left. e^{x+y}\right|_{(0,0)} = 1 \\
\frac{\partial^2 f}{\partial y^2}(\mathbf{a}) &= \left. e^{x+y}\right|_{(0,0)} = 1
\end{aligned}
$$

So

$$
\begin{aligned}
f(\mathbf{x}) &= f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x}-\mathbf{a}) + \frac{1}{2}(\mathbf{x}-\mathbf{a})^T \mathbf{H} f(\mathbf{a})(\mathbf{x}-\mathbf{a}) + \\
&= 1 + (1,1)\cdot(x,y) + (x \quad y)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \mathcal{O}\left(\|\mathbf{x}\|^3\right) \\
&= 1 + x + y + \frac{1}{2}(x \quad y)\begin{pmatrix} x+y \\ x+y \end{pmatrix} + \mathcal{O}\left(\|\mathbf{x}\|^3\right) \\
&= 1 + x + y + \frac{1}{2}\left(x^2 + xy + yx + y^2\right) + +\mathcal{O}\left(\|\mathbf{x}\|^3\right) \\
&= 1 + (x+y) + \frac{1}{2}(x+y)^2 + +\mathcal{O}\left(\|\mathbf{x}\|^3\right)
\end{aligned}
$$

▮

3.2.4. Determine the second order Taylor formula for $f(x,y) = \sin(xy) + \cos(xy)$ about $(0,0)$.

- We have

$$
\begin{aligned}
f(\mathbf{a}) &= \sin(0) + \cos(0) = 1 \\
\frac{\partial f}{\partial x}(\mathbf{a}) &= \left.(y\cos(xy) - y\sin(xy))\right|_{(0,0)} = 0 \\
\frac{\partial f}{\partial y}(\mathbf{a}) &= \left.(x\cos(xy) - x\sin(xy))\right|_{(0,0)} = 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x^2}(\mathbf{a}) &= \left.\left(-y^2\sin(xy) - y^2\cos(xy)\right)\right|_{(0,0)} = 0 \\
\frac{\partial^2 f}{\partial x \partial y}(\mathbf{a}) &= \left.(\cos(xy) - xy\sin(xy) - \sin(xy) - xy\cos(xy))\right|_{(0,0)} = 1 \\
\frac{\partial^2 f}{\partial y \partial x}(\mathbf{a}) &= \left.\cos(xy) - xy\sin(xy) - \sin(xy) - xy\cos(xy)\right|_{(0,0)} = 1 \\
\frac{\partial^2 f}{\partial y^2}(\mathbf{a}) &= \left.\left(-x^2\sin(xy) - x^2\cos(xy)\right)\right|_{(0,0)} = 0
\end{aligned}
$$

So

$$
\begin{aligned}
f(\mathbf{x}) &= f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x}-\mathbf{a}) + \frac{1}{2}(\mathbf{x}-\mathbf{a})^T \mathbf{H} f(\mathbf{a})(\mathbf{x}-\mathbf{a}) + \mathcal{O}\left(\|\mathbf{x}-\mathbf{a}\|^3\right) \\
&= 1 + (0,0)\cdot(x,y) + \frac{1}{2}(x \quad y)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + +\mathcal{O}\left(\|\mathbf{x}\|^3\right) \\
&= 1 + 0 + \frac{1}{2}(x \quad y)\begin{pmatrix} y \\ x \end{pmatrix} + +\mathcal{O}\left(\|\mathbf{x}\|^3\right) \\
&= 1 + \frac{1}{2}(xy + yx) \\
&= 1 + xy + +\mathcal{O}\left(\|\mathbf{x}\|^3\right)
\end{aligned}
$$

▮

Section 3.3

3.3.1. Find the critical points of the given function and then determine whether they are local maxima, local minima, or saddle points.

$$f(x, y) = x^2 - y^2 + xy$$

- In order to find the critical points of f we look for solutions of

$$\mathbf{0} = \nabla f(x, y)) = (2x + y, -2y + x) \quad .$$

We thus need to find solutions of

$$\begin{aligned} 2x + y &= 0 \\ x - 2y &= 0 \end{aligned}$$

Using the second equation to replace x by $2y$ in the first equation we obtain

$$4y + y = 0 \quad \Rightarrow \quad y = 0 \quad \Rightarrow \quad x = 2y = 0 \quad .$$

We thus have one critical point at $(0, 0)$. The Hessian matrix at (0,0) is

$$\mathbf{H}f(0,0) = \left. \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \right|_{(0,0)} = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$$

Since the determinant of $\mathbf{H}f(0,0)$

$$D = -4 - 1 = -5$$

is negative, the point (0,0) must be a saddle point of f. ∎

3.3.2. Find the critical points of the given function and then determine whether they are local maxima, local minima, or saddle points.

$$f(x, y) = x^2 + y^2 + 2xy$$

- In order to find the critical points of f we look for solutions of

$$\mathbf{0} = \nabla f(x, y)) = (2x + 2y, 2y + 2x) \quad .$$

The gradient ∇f will obviously vanish whenever $y = -x$. The Hessian matrix at a point (x,-x) will be

$$\mathbf{H}f(0,0) = \left. \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \right|_{(x,-x)} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

Since the determinant of $\mathbf{H}f(0,0)$

$$D = 4 - 4 = 0$$

vanishes. The Hessian test is thus inconclusive. Note however that if we write

$$f(x, y) = x^2 + 2xy + y^2 = (x + y)^2$$

we see that f is a non-negative function that attains the mininum value 0 along the line $y = -x$. Thus, every point of the line $y = -x$ is a local minimum of f. ∎

3.3.3. Find the critical points of the given function and then determine whether they are local maxima, local minima, or saddle points.

$$f(x, y) = e^{1 + x^2 - y^2}$$

- In order to find the critical points of f we look for solutions of

$$\mathbf{0} = \nabla f(x,y)) = \left(2xe^{1+x^2-y^2}, -2ye^{1+x^2-y^2}\right) \quad .$$

Since the exponential factor in each component never vanishes, we see that a point (x,y) will be a critical point of f if and only if

$$\begin{aligned} 2x &= 0 \\ 2y &= 0 \end{aligned}$$

We thus have one critical point at $(0,0)$. The Hessian matrix at (0,0) is

$$\begin{aligned} \mathbf{H}f(0,0) &= \left.\begin{pmatrix} \left(2+4x^2\right)e^{1+x^2-y^2} & (-4xy)\,e^{1+x^2-y^2} \\ (-4xy)\,e^{1+x^2-y^2} & \left(-2+4y^2\right)e^{1+x^2-y^2} \end{pmatrix}\right|_{(0,0)} \\ &= \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \end{aligned}$$

Since the determinant of $\mathbf{H}f(0,0)$

$$D = -4 - 0 = -4$$

is negative, the point (0,0) must be a saddle point of f. ∎

3.3.4. Find the critical points of the given function and then determine whether they are local maxima, local minima, or saddle points.

$$f(x,y) = 3x^2 + 2xy + 2x + y^2 + y + 4$$

- In order to find the critical points of f we look for solutions of

$$\mathbf{0} = \nabla f(x,y)) = (6x + 2y + 2, 2x + 2y + 1)$$

or

$$\begin{aligned} 6x + 2y &= -2 \\ 2x + 2y &= -1 \end{aligned}$$

Subtracting the second equation from the first we find

$$4x = -1 \qquad \Rightarrow \qquad x = -\frac{1}{4}$$

Inserting this result into the second equation yields

$$-\frac{1}{2} + 2y = -1 \qquad y = -\frac{1}{4}$$

We thus have one critical point at $\left(-\frac{1}{4}, -\frac{1}{4}\right)$. The Hessian matrix at this point is

$$\mathbf{H}f\left(-\frac{1}{4}, -\frac{1}{4}\right) = \left.\begin{pmatrix} 6 & 2 \\ 2 & 2 \end{pmatrix}\right|_{\left(-\frac{1}{4},-\frac{1}{4}\right)} = \begin{pmatrix} 6 & 2 \\ 2 & 2 \end{pmatrix}$$

Since the determinant of $\mathbf{H}f(0,0)$

$$D = (6)(2) - (2)(2) = 8$$

is positive, the point $\left(-\frac{1}{4}, -\frac{1}{4}\right)$ must be a local extremum. Since

$$\frac{\partial^2 f}{\partial x^2} = 6 > 0$$

we conclude that f has a local minimum at $\left(-\frac{1}{4}, -\frac{1}{4}\right)$. ∎

3.3.5. An examination of the function $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto (y - 3x^2)(y - x^2)$ will give an idea of the difficulty of finding conditions that guarantee that a critical point is a relative extrenum when Theorem 5 fails. Show that

(a) the origin is a critical point of f;

- We have

$$f(x, y) = y^2 - 4x^2 y + 3x^4$$

so

$$\nabla f = (12x^3 - 8xy, -4x^2 + 2y) \quad .$$

Since ∇f vanishes at the point (0,0) the origin is a critical of f. ▮

(b) f has a relative minimum at (0,0) on every straight line through (0,0); that is, if $g(t) = (at, bt)$, then $f \circ g : \mathbb{R} \to \mathbb{R}$ has a relative minimum at 0, for every choice of a and b;

- We have

$$f \circ g(t) = f(g(t)) = b^2 t^2 - 4a^2 b t^3 + 3a^4 t^4$$

Differentiating with respect to t we find

$$\left.\frac{d}{dt} f \circ g\right|_{t=0} = \left.(2b^2 t - 12a^2 b t^2 + 12a^4 t^3)\right|_{t=0} = 0$$

and so $f \circ g$ has a critical point at $t = 0$. To see that $t = 0$ is a local minimum, we apply the second derivative test.

$$\left.\frac{d^2}{dt^2} f \circ g\right|_{t=0} = \left.(2b^2 - 24a^2 b t + 36a^4 t^2)\right|_{t=0} = 2b^2 \quad .$$

Thus the second derivative is positive so long as $b \neq 0$. From this we can conclude that f attains a local minimum at $t = 0$ along any line $g(t) = (at, bt)$ with $b \neq 0$. Along lines of the form $g(t) = (at, 0)$, we have

$$f \circ g(t) = 3a^4 t^4 \quad .$$

This function obviously attains a local minimum at $t = 0$. Thus, along any line through (0,0) f attains a local minimum. ▮

(c) The origin is not a relative minimum of f.

- Consider the values of f along the curve $\sigma(t) = (t, 2t)$. We have

$$f(\sigma(t)) = (4t^4 - 8t^4 + 3t^4) = -t^4 \quad .$$

The composed function $f(\sigma(t))$ obviously has a local maximum at $t = 0$. Thus, if we procede away from the origin along this particular curve the value of f actually decreases. Since The value of f increases or decreases depending on the way in which we away from the origin we conclude that the origin can not be a local extremum of f. ▮

3.3.6. Let $f(x, y) = x^2 - 2xy + y^2$. Here $D = 0$. Can you say whether the critical points are local minima, local maxima, or saddle points?

- We have

$$\nabla f = (2x - 2y, -2x + 2y)$$

and so every point on the line $y = x$ will be a critical point of f. If we try to apply the second derivative test, we find that the determinant of the Hessian matrix

$$\mathbf{H}f(x,y) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

vanishes for all (x,y). Thus, the second derivative test will be inclusive. However, if we write

$$f(x,y) = x^2 - 2xy - y^2 = (x-y)^2$$

it is obvious that f attains a local minimum along the line $y = x$. ∎

Section 3.4

3.4.1. Find the extrema of $f(x,y,z) = x - y + z$ subject to the constraint $x^2 + y^2 + z^2 = 2$.

- Set

$$g(x,y,z) = x^2 + y^2 + z^2$$

We must look for solutions (x,y,z,λ) of

$$\begin{aligned} x^2 + y^2 + z^2 &= 2 \\ \nabla f(x,y,z) &= \lambda \nabla g(x,y,z) \end{aligned}$$

The second (vector) equation is equivalent to

$$\begin{aligned} 1 = \frac{\partial f}{\partial x} &= \lambda \frac{\partial g}{\partial x} = 2\lambda x \\ -1 = \frac{\partial f}{\partial y} &= \lambda \frac{\partial g}{\partial y} = 2\lambda y \\ 1 = \frac{\partial f}{\partial z} &= \lambda \frac{\partial g}{\partial z} = 2\lambda z \end{aligned}$$

so

$$x = \frac{1}{2\lambda} \quad , \quad y = -\frac{1}{2\lambda} \quad , \quad z = \frac{1}{2\lambda} \quad .$$

Inserting these expressions for x, y, z into the contraint equation yields

$$2 = \left(\frac{1}{2\lambda}\right)^2 + \left(-\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = \frac{3}{4\lambda^2}$$

or

$$\lambda = \pm\sqrt{\frac{3}{8}} \qquad \Rightarrow \qquad x = \pm\sqrt{\frac{2}{3}} \quad , \quad y = \mp\sqrt{\frac{2}{3}} \quad , \quad z = \pm\sqrt{\frac{2}{3}}$$

We thus have two critical points $\left(\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$ and $\left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}\right)$. The value of f at these points is

$$\begin{aligned} f\left(\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right) &= \sqrt{\frac{2}{3}} \\ f\left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}\right) &= -\sqrt{\frac{2}{3}} \end{aligned}$$

The first critical point is therefore (likely to be) a local maximum and the the second critical point is therefore (likely to be) a local minimum. That that this conclusion is in fact true - from Theorem 6 on page 260, we know that a continuous function on a closed subset of $\mathbb{R}^n$ always attains a maximum and a minimum. Since the surface $x^2+y^2+z^2 = 2$ is closed, f must have a minimum and a maximum on this surface. Since we discovered only two critical points, one must be the minimum of f on this surface and the other must be the maximum of f on this surface. ∎

3.4.2. Find the extrema of $f(x,y) = x$ subject to the constraint $x^2 + 2y^2 = 3$.

- The critical points of f on the ellipse

$$g(x,y) = x^2 + 2y^2 = 3$$

must be solutions of

$$\begin{aligned} \nabla f &= \lambda \nabla g \\ g(x,y) &= 3 \end{aligned}$$

The first equation is equivalent to

$$\begin{aligned} 1 &= 2\lambda x \\ 0 &= 4\lambda y \end{aligned}$$

We must therefore have

$$x = \frac{1}{2\lambda} \quad , \quad y = 0 \quad .$$

But if $y = 0$, the constraint $g(x, y) = 3$ implies

$$x^2 = 3 \qquad \Rightarrow \qquad x = \pm\sqrt{3} \quad .$$

We thus have two extrema of $f(x, y) = x$ on the ellipse $x^2 + 2y^2 = 3$; namely, $\left(\sqrt{3}, 0\right)$ and $\left(-\sqrt{3}, 0\right)$. Since the ellipse $x^2 + 2y^2 = 3$ is a closed subset of $\mathbb{R}^2$, f must obtain both an absolute maximum and an absolute minimum on this curve (by Theorem 6). Since

$$f\left(\sqrt{3}, 0\right) = \sqrt{3} \quad > \quad -\sqrt{3} = f\left(-\sqrt{3}, 0\right)$$

we conclude that $\left(\sqrt{3}, 0\right)$ is the point on the ellipse $x^2+2y^2 = 3$ at which f is maximized and $\left(-\sqrt{3}, 0\right)$ is the point at which f is minimized. ∎

3.4.3. Find the extrema of $f(x, y) = 3x + 2y$ subject to the constraint $2x^2 + 3y^2 = 3$.

- The critical points of f on the ellipse

$$g(x, y) = 2x^2 + 3y^2 = 3$$

must be solutions of

$$\begin{aligned} \nabla f &= \lambda \nabla g \\ g(x, y) &= 3 \end{aligned}$$

The first equation is equivalent to

$$\begin{aligned} 3 &= 4\lambda x \\ 2 &= 6\lambda y \end{aligned}$$

We must therefore have

$$x = \frac{3}{4\lambda} \quad , \quad y = \frac{1}{3\lambda} \quad .$$

But the constraint $g(x, y) = 3$ implies

$$2\left(\frac{3}{4\lambda}\right)^2 + 3\left(\frac{1}{3\lambda}\right)^2 = 3 \qquad \Rightarrow \qquad \frac{1}{\lambda^2}\left(\frac{9}{8} + \frac{1}{3}\right) = 3 \qquad \Rightarrow \qquad \lambda = \pm\sqrt{\frac{35}{72}}.$$

But then

$$x = \pm\frac{3}{4}\sqrt{\frac{72}{35}} = \pm\frac{9}{\sqrt{70}} \qquad , \qquad y = \pm\frac{1}{3}\sqrt{\frac{72}{35}} = \pm\sqrt{\frac{8}{35}} = \pm\frac{4}{\sqrt{70}} \quad .$$

The critical points of f on the ellipse $2x^2 + 3y^2 = 3$ are thus $\left(\pm\frac{9}{\sqrt{70}}, \pm\frac{4}{\sqrt{70}}\right)$. Since the ellipse $2x^2 + 3y^2 = 3$ is a closed bounded subset of $\mathbb{R}^2$, one of these points must be a maximum and the other must be a minimum (see Theorem 6 on page 260). Evaluating f at these two points we see

$$\begin{aligned} f\left(\frac{9}{\sqrt{70}}, \frac{4}{\sqrt{70}}\right) &= \frac{35}{\sqrt{70}} \\ f\left(-\frac{9}{\sqrt{70}}, -\frac{4}{\sqrt{70}}\right) &= -\frac{35}{\sqrt{70}} \end{aligned}$$

we conclude that $\left(\frac{9}{\sqrt{70}}, \frac{4}{\sqrt{70}}\right)$ is the maximum of $3x+2y$ on the ellipse $2x^2+3y^2 = 3$ and $\left(-\frac{9}{\sqrt{70}}, -\frac{4}{\sqrt{70}}\right)$ is the minimum of $3x + 2y$ on the ellipse $2x^3 + 3y^2 = 3$ ∎

Section 3.5

3.5.1.* Let $F(x, y) = 0$ define a curve in the xy plane through the point (x_o, y_o). Assume that $(\partial F/\partial y)(x_o, y_o) \neq 0$. Show that this curve can be locally represented by the graph of a function $y = g(x)$. Show that the line orthogonal to $\nabla F(x_o, y_o)$ agrees with the tangent line to the graph of $y = g(x)$.

- ∎

4.**3.5.2.*** (a) Check directly (i.e., without using Theorem 10) where we can solve $F(x, y) = y^2 + y + 3x + 1 = 0$ for y in terms of x.

- ∎

(b) Check that your answer in part (a) agrees with the answer you expect from the implicit function theorem. Compute dy/dx.

- ∎

3.5.3.* Show that $x^3z^2 - z^3yx = 0$ is solvable for z as a function of (x, y) near (1,1,1), but not near the origin. Compute $\partial z/\partial x$ and $\partial z/\partial y$ at (1,1).

- ∎