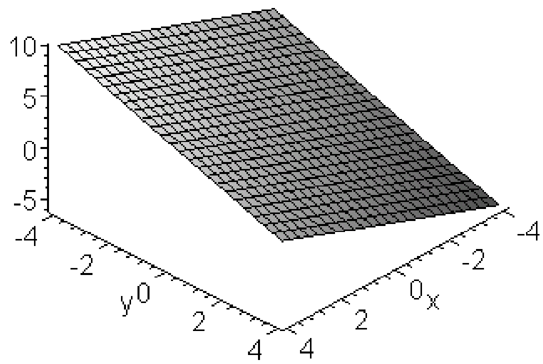
Math 4013
Solutions to Homework Problems from Chapter 2

Section 2.1

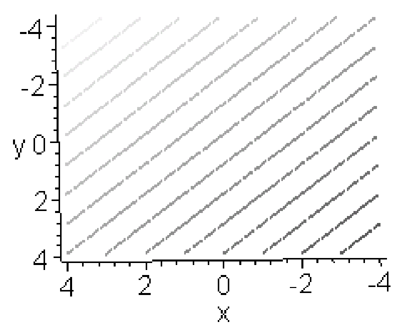
1. Sketch the level curves and graphs of the following functions:
   (a)

$$f:\mathbb{R}^2 \rightarrow \mathbb{R} \quad , \quad (x,y) \mapsto x-y+2$$



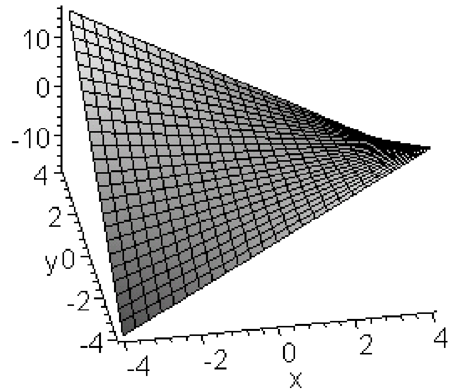
The level curves are just the lines

$$x-y+2=C \qquad \Rightarrow \quad y=x+2-C$$



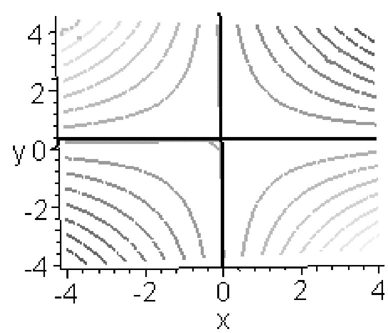
(b)

$$f:\mathbb{R}^2 \rightarrow \mathbb{R} \quad , \quad (x,y) \mapsto -xy$$



The level curves are the curves where

$$-xy = C \qquad \Rightarrow \quad y = -\frac{C}{x}$$



2. (a) Describe the behavior, as c varies, of the level curve $f(x, y) = c$ for the function

$$f(x, y) = x^2 + y^2 + 1 \quad .$$

- The equation of the defining the level curves

$$f(x, y) = c$$

is just

$$x^2 + y^2 + 1 = c$$

or

$$x^2 + y^2 = 1 - c \quad .$$

This is the equation of circle about the origin with radius $\sqrt{1 - c}$. Note that c must be less than or equal to one; otherwise the equation of the level curve has no solutions for real numbers x and y. Below we sketch a few level curves. ∎

3. Sketch or describe the level surfaces and a section of the graph of the following function:

$$f : \mathbb{R}^3 \to \mathbb{R} \, ; \quad (x, y, z) \mapsto -x^2 - y^2 - z^2 \quad .$$

- The level surfaces of this function must be the solution sets of equations of the form

$$f(x, y, z) = c \quad ,$$

or

$$-x^2 - y^2 - z^2 = c \quad ,$$

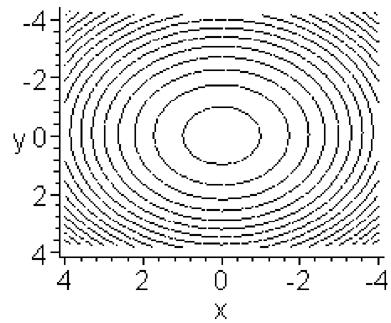


FIGURE 1

or

$$x^2 + y^2 + z^2 = -c \quad .$$

The last equation is just the equation of a sphere about the origin in $\mathbb{R}^3$ of radius $\sqrt{-c}$. We conclude that the level surfaces of f are just sphere's about the origin.
The $z = 0$ section of the graph of f will consist of points in $\mathbb{R}^4$ corresponding to the intersection of the graph of f with the plane $z = 0$:

$$\begin{aligned} S &= \left\{(x, y, z, t) \in \mathbb{R}^4 \mid t = -x^2 - y^2 - z^2\right\} \cap \left\{(x, y, z, t) \in \mathbb{R}^4 \mid z = 0\right\} \\ &= \left\{(x, y, z, t) \in \mathbb{R}^4 \mid z = 0 \,, \quad t = -x^2 - y^2\right\} \end{aligned}$$

▮

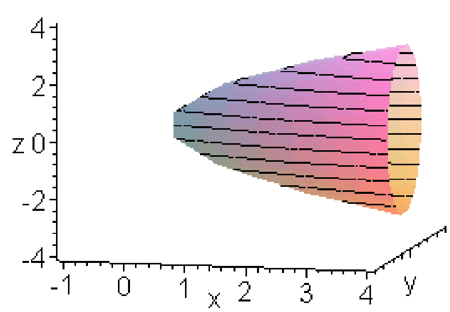
4. Sketch or describe the surface in $\mathbb{R}^3$ corresponding to the equation

$$4x^2 + y^2 = 16 \quad .$$

   - This surface will just be a two dimensional ellipse extende to $\pm\infty$ in the z-direction; a squashed cylinder if you will. ▮

5. Sketch or describe the surface in $\mathbb{R}^3$ corresponding to the equation

$$\frac{x}{4} = \frac{y^2}{4} + \frac{z^2}{9} \quad .$$



Section 2.2

1. Show that the following subset of $\mathbb{R}^2$ is open.

$$B = \left\{(x, y) \in \mathbb{R}^2 \mid y > 0\right\} \quad .$$

   - By definition, a subset B of $\mathbb{R}^2$ is open if for any point $\mathbf{r} \in B$ there exists a disk $D_\rho(\mathbf{r})$ of radius $\rho > 0$ about $\mathbf{r}$ that lies entirely within B.

– * Let $\mathbf{r} = (x, y)$ be an arbitrary point of B. Consider the disk

$$D_{\frac{y}{2}}(\mathbf{r}) = \left\{ \mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| < \rho/2 \right\} \quad .$$

Clearly, this disk lies completely with B. This this construction works for any point $\mathbf{r} \in B$, we conclude that for every point of B there is an open disk containing that point lying completely within B. Thus, B is open. ∎

2. Prove that if U and V are neighborhoods of $\mathbf{x} \in \mathbb{R}^n$, then so are $U \cup V$ and $V \cap U$.
   - By definition, a set S is a neighborhood of a point $\mathbf{x} \in \mathbb{R}^n$ if
     (i) $\mathbf{x} \in S$;
     (ii) S is open.
     Clearly, since U and V are neighborhoods of $\mathbf{x}$, $\mathbf{x} \in U$ and $\mathbf{x} \in V$ and so $\mathbf{x} \in U \cup V$ and $\mathbf{x} \in U \cap V$. It thus remains to show that both $U \cup V$ and $U \cap V$ are open.
     We first prove that $U \cup V$ is open. Let $\mathbf{y} \in U \cup V$. Then either
     (i) $\mathbf{y} \in U$
     (ii) $\mathbf{y} \in V$ (Note that these two cases are exhaustive but not exclusive.) In the first case, since U is open (since it is a neighborhood), there must exist an open ball $B_{\rho_1}(\mathbf{y})$ about $\mathbf{y}$ completely contained in U. But then this ball is also completely contained in $U \cup V$. In the second case, since V is open, there must exist some open ball $B_{\rho_2}(\mathbf{y})$ completely contained in V and so completely contained in $U \cup V$. Thus, in either case there must exist an open ball about $\mathbf{y}$ that is completely contained in $U \cup V$. Hence, $U \cup V$ is open.
     We now show that $U \cap V$ is open. Let $\mathbf{y} \in U \cap V$. Clearly, $\mathbf{y} \in U$ and $\mathbf{y} \in V$. Since U and V are open, there must exist open balls

$$\begin{aligned} B_{\rho_1}(\mathbf{y}) &\subset U \\ B_{\rho_2}(\mathbf{y}) &\subset V \end{aligned}$$

     Set

$$\rho = min\,[\rho_1, \rho_2] \quad .$$

     Then $B_\rho(\mathbf{y}) \subset B_{\rho_1}(\mathbf{y}) \subset U$ and $B_\rho(\mathbf{y}) \subset B_{\rho_2}(\mathbf{y}) \subset V$, and so

$$B_\rho(\mathbf{y}) \subset U \cap V \quad .$$

     There thus exists an open ball about $\mathbf{y}$ that is completely contained in $U \cap V$. Since $\mathbf{y}$ is an arbitrary point of $U \cap V$, we conclude that $U \cap V$ must be an open set. ∎

3. Compute the following limits. (The text allows the reader to assume that the exponential, sine, and cosine functions are all continuous. By Example 7 on page 108, any polynomial function on $\mathbb{R}^n$ is also continuous.)
   (a)

$$\lim_{(x,y)\to(0,1)} x^3 y$$

   - Since $x^3 y$ is a polynomial function on $\mathbb{R}^2$, it is continuous. Hence, its limit at a point $\mathbf{r} \in \mathbb{R}^2$ coincides with its value at the point $\mathbf{r}$. Thus

$$\lim_{(x,y)\to(0,1)} x^3 y = x^3 y \Big|_{(0,1)} = 0$$

     ∎

   (b)

$$\lim_{(x,y)\to(0,1)} e^x y$$

   - The exponential function $f(x, y) = e^x$ and the linear function $g(x, y) = y$ are both continuous for all $(x, y) \in R^2$. By Theorem 4(iii) so must be $fg(x, y) = f(x, y)g(x, y) = e^x y$. Since $e^x y$ is continuous, its limit at any point must coincide with its value there. Thus,

$$\lim_{(x,y)\to(0,1)} e^x y = e^x y \Big|_{(0,1)} = 1 \quad .$$

     ∎

(c)

$$\lim_{x\to 0} \frac{\sin^2(x)}{x}$$

- The function in the denominator vanishes at $x = 0$ and so we cannot apply continuity arguments to evaluate the limit. (In other words, Theorem 4 (iv) cannot be applied in this case.) We can, however, evaluate this limit using l'Hospital's rule: if f and g are continuous differentiable functions and

$$\lim_{x\to x_o} f(x) = 0 = \lim_{x\to x_o} g(x)$$

then

$$\lim_{x\to x_o} \frac{f(x)}{g(x)} = \lim_{x\to x_o} \frac{f'(x)}{g'(x)} \quad .$$

Applying l'Hospital's rule to the case at hand we find

$$\begin{aligned} \lim_{x\to 0} \frac{\sin^2(x)}{x} &= \lim_{x\to 0} \frac{2\cos(x)\sin(x)}{1} \\ &= \lim_{x\to 0} 2\cos(x)\sin(x) \\ &= 2\cos(x)\sin(x)\,|_{x=0} \\ &= 0 \end{aligned}$$

In the last couple of steps we have used the fact that the sine and cosine functions are continuous - so their limits were taken by simply evaluated them at the limit point. ∎

(d)

$$\lim_{x\to 0} \frac{\sin^2(x)}{x^2}$$

- This problem is of course similar to the preceding one. The only difference is that we must apply l'Hospital's Rule twice before we get an obviously continuous function.

$$\begin{aligned} \lim_{x\to 0} \frac{\sin^2(x)}{x^2} &= \lim_{x\to 0} \frac{2\cos(x)\sin(x)}{2x} \\ &= \lim_{x\to 0} \frac{-2\sin^2(x) + 2\cos^2(x)}{2} \\ &= \lim_{x\to 0} \left(-\sin^2(x) + \cos^2(x)\right) \\ &= 1 \end{aligned}$$

∎

4. Compute the following limits if they exist.

(a)

$$\lim_{(x,y)\to(0,0)} \left(x^2 + y^2 + 3\right)$$

- This is the limit of a polynomial function on $\mathbb{R}^2$. Since polynomial functions are continuous, we can evaluate the limit by simply evaluating the polynomial at the limit point. Thus,

$$\lim_{(x,y)\to(0,0)} \left(x^2 + y^2 + 3\right) = \left(x^2 + y^2 + 3\right)\Big|_{(0,0)} = 3 \quad .$$

∎

(b)

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2 + 2}$$

- Both $f(x,y) = xy$ and $g(x,y) = x^2 + y^2 + 2$ are continuous for all x. Moreover, $g(0,0) = 2 \neq 0$, so we can first apply Theorem 4 (iv) to conclude that $\frac{1}{g}$ is continuous at (0,0) and then Theorem 4 (iii) to conclude that $\frac{f}{g} = \frac{xy}{x^2+y^2+2}$ is continuous at (0,0). Therefore, the limit at $(0,0)$ must coincide with evaluation at (0,0). Thus,

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2+2} = \left.\frac{xy}{x^2+y^2+2}\right|_{(0,0)} = 0 \quad .$$

∎

(c)

$$\lim_{(x,y)\to(0,0)} \frac{e^{xy}}{x+1}$$

- This is similar to the preceding problem. Since $f(x,y) = e^{xy}$ is continuous at (0,0) and $g(x,y) = x+1$ is continuous and non-vanishing at (0,0), f/g is continuous at (0,0) and so we can evaluate its limit at (0,0) by evaluating f/g at (0,0):

$$\lim_{(x,y)\to(0,0)} \frac{e^{xy}}{x+1} = \left.\frac{e^{xy}}{x+1}\right|_{(0,0)} = \frac{1}{1} = 1 \quad .$$

∎

(d)

$$\lim_{(x,y)\to(0,0)} \frac{\cos(x) - 1 - \frac{x^2}{2}}{x^4+y^4}$$

- In this case, the limit does not exist. To see this, consider the following curve through (0,0):

$$\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2 \quad , \quad t \longmapsto (0,t) \quad ,$$

and set

$$f(x,y) = \frac{\cos(x) - 1 - \frac{x^2}{2}}{x^4+y^4} \quad .$$

If the limit of $f(x,y)$ as (x,y) approaches (0,0) exists, it must be coincide with

$$\lim_{t\to 0} f(\gamma(t)) = \lim_{t\to 0} \frac{-1}{t^4}$$

The right hand side, however, is undefined. We conclude that the limit

$$\lim_{(x,y)\to(0,0)} \frac{\cos(x) - 1 - \frac{x^2}{2}}{x^4+y^4}$$

does not exist. ∎

(e)

$$\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2}$$

- This limit does not exist either. To see this, consider the following two curves through the (0,0):

$$\begin{aligned} \gamma_1 &: \mathbb{R} \longrightarrow \mathbb{R}^2 \quad , \quad t \longmapsto (t,0) \\ \gamma_2 &: \mathbb{R} \longrightarrow \mathbb{R}^2 \quad , \quad t \longmapsto (t,t) \end{aligned}$$

and set

$$f(x,y) = \frac{(x-y)^2}{x^2+y^2} \quad .$$

If the limit (e) is to exist, we must have

$$\lim_{t\to 0} f(\gamma_1(t)) = \lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2} = \lim_{t\to 0} f(\gamma_2(t))$$

Evaluating the limit on the far left hand side we find

$$\lim_{t\to 0} f(\gamma_1(t)) = \lim_{t\to 0} \frac{(t-0)^2}{t^2+0^2} = 1 \quad ,$$

but the limit on the far right hand side is

$$\lim_{t\to 0} f(\gamma_2(t)) = \lim_{t\to 0} \frac{(t-t)^2}{t^2+t^2} = 0 \quad .$$

Since these two limits do not agree, we can conclude the limit

$$\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2}$$

does not exist. ∎

5. Show that the map

$$f:\mathbb{R}\to\mathbb{R}, \quad x \;\mapsto \frac{x^2e^x}{2-\sin(x)}$$

is continuous.

- We can prove this using the Theorem 4 and the fact that the exponential function, the sine function, and any polynomial function is continuous.
  Since x^2 and e^x are continuous, so is their product $p(x) = x^2e^x$ by Theorem 4 (iii). Since the constant function $h(x) = 2$ and $\sin(x)$ are continuous, so is their sum $q(x) = 2 - \sin(x)$, by Theorem 4 (ii). Since $2-\sin(x)$ is nowhere zero, the quotient function $1/q(x)$ is continous everywhere, by Theorem 4 (iv). Finally since both $p(x)$ and $1/q(x)$ are continuous, so is

$$p(x)\cdot\frac{1}{q(x)} = \frac{x^2e^x}{2-\sin(x)}$$

by Theorem 4 (iii). ∎

Section 2.3

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for each of the following functions.
   (a) $f(x,y) = xy$.
   -
$$\begin{aligned}\frac{\partial f}{\partial x} &= y\\ \frac{\partial f}{\partial y} &= x\end{aligned}$$

∎

   (b) $f(x,y) = x\cos(x)\cos(y)$.
   -
$$\begin{aligned}\frac{\partial f}{\partial x} &= \cos(x)\cos(y) - x\sin(x)\cos(y)\\ \frac{\partial f}{\partial y} &= -x\cos(x)\sin(y)\end{aligned}$$

∎

2. Evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the function $z = \log\left[\sqrt{1+xy}\right]$ at the points (1,2) and (0,0).
   -
$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{1}{\sqrt{1+xy}}\left(\frac{y}{2\sqrt{1+xy}}\right) = \frac{y}{2\,(1+xy)}\\ \frac{\partial z}{\partial y} &= \frac{1}{\sqrt{1+xy}}\left(\frac{x}{2\sqrt{1+xy}}\right) = \frac{x}{2\,(1+xy)}\end{aligned}$$

Thus,

$$\begin{aligned}
\left.\frac{\partial z}{\partial x}\right|_{(1,2)} &= \frac{2}{2(1+2)} = \frac{1}{3} \\
\left.\frac{\partial z}{\partial y}\right|_{(1,2)} &= \frac{1}{2(1+2)} = \frac{1}{6} \\
\left.\frac{\partial z}{\partial x}\right|_{(0,0)} &= \frac{0}{2(1+0)} = 0 \\
\left.\frac{\partial z}{\partial y}\right|_{(0,0)} &= \frac{0}{2(1+0)} = 0
\end{aligned}$$

∎

3. Find the partial derivatives $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ when $w = xe^{x^2+y^2}$.
   - $$\begin{aligned}
\frac{\partial w}{\partial x} &= e^{x^2+y^2} + x\left(2xe^{x^2+y^2}\right) \\
\frac{\partial w}{\partial y} &= x\left(2ye^{x^2+y^2}\right)
\end{aligned}$$

   ∎

4. Show that the function
$$f(x,y) = \frac{2xy}{\left(x^2+y^2\right)^2}$$
is differentiable at each point in its domain. Is this function C^1.
   - The natural domain of $f(x,y)$ is $\mathbb{R}^2 - \{(0,0)\}$ (the natural domain of a rational function in n variables is $\mathbb{R}^n$ minus the points where the denominator vanishes). Now
$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{2y\left(x^2+y^2\right)^2 - 2xy\left(2\left(x^2+y^2\right)2x\right)}{\left(x^2+y^2\right)^4} \\
&= \frac{2x^4y + 4x^2y^3 + 2y^5 - 8x^4y - 8x^2y^3}{\left(x^2+y^2\right)^4} \\
\frac{\partial f}{\partial y} &= \frac{2x\left(x^2+y^2\right)^2 - 2xy\left(2\left(x^2+y^2\right)2y\right)}{\left(x^2+y^2\right)^4} \\
&= \frac{2x^5 + 4x^3y^2 + 2xy^4 - 8x^3y^2 - 8xy^4}{\left(x^2+y^2\right)^4}
\end{aligned}$$
Both partials are rational functions with singularities at the origin. They are therefore continuous everywhere except possibly at (0,0). Since the partial derivatives are continuous everywhere within the domain of f, f is C^1 throughout its domain. ∎

5. Find the equation of the plane tangent to the surface $z = x^2 + y^3$ at (3,1,10).
   - The tangent plane to a graph $z = f(x,y)$ at the point $(x_o, y_o, f(x_o, y_o))$ consists of points $(x,y,z) \in \mathbb{R}^3$ satisfying the following equation
$$z = f(x_o, y_o) + \left.\frac{\partial f}{\partial x}\right|_{(x_o,y_o)} (x - x_o) + \left.\frac{\partial f}{\partial y}\right|_{(x_o,y_o)} (y - y_o) \quad .$$
For the case at hand,
$$\begin{aligned}
\frac{\partial f}{\partial x} &= 2x \\
\frac{\partial f}{\partial y} &= 3y
\end{aligned}$$
and so we must have
$$z = 10 + (6)(x-3) + (3)(y-1)$$

or

$$6x+3y-z=11 \quad .$$

▮

6. Compute the matrix of partial derivatives of the following function:

$$f:\mathbb{R}^2 \to \mathbb{R}^3,\ f(x,y)=(xe^y+\cos(y),x,x+e^y) \quad .$$

•

$$Df=\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{pmatrix}=\begin{pmatrix} e^y & xe^y-\sin(y) \\ 1 & 0 \\ 1 & e^y \end{pmatrix}$$

▮

7. Find the equation of the tangent plane to $z=x^2+2y^3$ at (1,1,3).

• We have

$$\begin{aligned} \frac{\partial z}{\partial x} &= 2x \\ \frac{\partial z}{\partial y} &= 6y^2 \end{aligned}$$

and so (using the formula appearing in the solution to Problem 2.3.5), the equation of the tangent plane to the graph of x^2+2y^3 at (1,1,3) is

$$z=3+(2)(x-1)+(6)(y-1)$$

or

$$2x+6y-z=5 \quad .$$

▮

Section 2.4

1. Find $\sigma'(t)$ and $\sigma'(0)$ for the following path.

$$\sigma(t)=\left(e^t,\cos(t),\sin(t)\right).$$

• We have

$$\begin{aligned} \sigma'(t) &= \frac{d\sigma}{dt}(t) \\ &= \left(\frac{d}{dt}\left(e^t\right),\frac{d}{dt}\left(\cos(t)\right),\frac{d}{dt}\left(\sin(t)\right)\right) \\ &= \left(e^t,-\sin(t),\cos(t)\right) \end{aligned}$$

and so

$$\sigma'(0)=\left(e^0,-\sin(0),\cos(0)\right)=(1,0,1)$$

▮

2. Determine the velocity and acceleration vectors, and the equation of the tangent line for each of the following curves at the specified value of t.

(a) $\mathbf{r}(t)=6t\mathbf{i}+3t^2\mathbf{j}+t^3\mathbf{k}$, $t=0$

•

$$\begin{aligned} \frac{d\mathbf{r}}{dt} &= 6\mathbf{i}+6t\mathbf{j}+3t^2\mathbf{k} \\ \frac{d^2\mathbf{r}}{dt^2} &= 6\mathbf{j}+6t\mathbf{k} \end{aligned}$$

At $t=0$, we have

$$\begin{aligned} \mathbf{r}(0) &= 0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=\mathbf{0} \\ \mathbf{r}'(0) &= 6\mathbf{i} \end{aligned}$$

Therefore, the equation of the tangent line at $t = 0$ is

$$\mathbf{l}(t) = \mathbf{r}(0) + t\mathbf{r}'(0) = 6t\mathbf{i} \quad .$$

∎

(b) $\sigma(t) = \left(\sin(3t), \cos(3t), 2t^{3/2}\right)$, $t = 1$

• We have

$$\begin{aligned} \sigma'(t) &= \left(3\cos(3t), -3\sin(3t), 3t^{1/2}\right) \\ \sigma''(t) &= \left(-9\sin(3t), -9\cos(3t), \frac{3}{2}t^{-1/2}\right) \quad . \end{aligned}$$

At $t = 1$ we have

$$\begin{aligned} \sigma(1) &= (\sin(3), \cos(3), 2) \\ \sigma'(1) &= (3\cos(3), -3\sin(3), 3) \end{aligned}$$

so the equation of the tangent line at $t = 1$ is

$$\mathbf{l}(t) = \sigma(1) + t\sigma'(1) = (\sin(3) + 3\cos(3)t, \cos(3) - 3\sin(3)t, 2 + 3t)$$

∎

(c) $\sigma(t) = \left(\cos^2(t), 3t - t^3, t\right)$, $t = 0$

• We have

$$\begin{aligned} \sigma'(t) &= \left(2\cos(t)\sin(t), 3 - 3t^2, 1\right) \\ \sigma''(t) &= \left(-2\sin^2(t) + 2\cos^2(t), -6t, 0\right) \quad . \end{aligned}$$

At $t = 0$ we have

$$\begin{aligned} \sigma(0) &= (1, 0, 0) \\ \sigma'(t) &= (0, 3, 1) \end{aligned}$$

so the equation of the tangent line at $t = 1$ is

$$\mathbf{l}(t) = \sigma(0) + t\sigma'(0) = (1, 3t, t)$$

∎

(d) $\sigma(t) = (0, 0, t)$, $t = 1$

• We have

$$\begin{aligned} \sigma'(t) &= (0, 0, 1) \\ \sigma''(t) &= (0, 0, 0) \quad . \end{aligned}$$

At $t = 1$ we have

$$\begin{aligned} \sigma(0) &= (0, 0, 1) \\ \sigma'(t) &= (0, 0, 1) \end{aligned}$$

so the equation of the tangent line at $t = 1$ is

$$\mathbf{l}(t) = \sigma(1) + t\sigma'(1) = (0, 0, 1 + t) \quad .$$

∎

3. Determine the velocity and acceleration vectors, and the equation of the tangent line for each of the following curves at the specified value of t.

(a) $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$, $t = 0$

•

$$\begin{aligned} \frac{d\mathbf{r}}{dt} &= -\sin(t)\mathbf{i} + \cos(t)\mathbf{j} \\ \frac{d^2\mathbf{r}}{dt^2} &= -\cos(t)\mathbf{i} - \sin(t)\mathbf{j} \end{aligned}$$

At $t = 0$, we have

$$\begin{aligned} \mathbf{r}(0) &= 1\mathbf{i} + 0\mathbf{j} = \mathbf{i} \\ \mathbf{r}'(0) &= 0\mathbf{i} + 1\mathbf{j} = \mathbf{j} \end{aligned}$$

Therefore, the equation of the tangent line at $t = 0$ is

$$\mathbf{l}(t) = \mathbf{r}(0) + t\mathbf{r}'(0) = \mathbf{i} + t\mathbf{j} \quad .$$

▮

(b) $\sigma(t) = \left(t\sin(t), t\cos(t), \sqrt{3}t\right)$, $t = 0$

- We have

$$\begin{aligned} \sigma'(t) &= \left(\sin(t) + t\cos(t), \cos(t) - t\sin(t), \sqrt{3}\right) \\ \sigma''(t) &= (2\cos(t) - t\sin(t), 2\sin(t) - t\cos(t), 0) \quad . \end{aligned}$$

At $t = 0$ we have

$$\begin{aligned} \sigma(0) &= (0, 1, 0) \\ \sigma'(0) &= \left(2, 0, \sqrt{3}\right) \end{aligned}$$

so the equation of the tangent line at $t = 0$ is

$$\mathbf{l}(t) = \sigma(0) + t\sigma'(0) = \left(2t, 1, \sqrt{3}t\right)$$

▮

(c) $\mathbf{r}(t) = \sqrt{2}t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}$, $t = 0$

- We have

$$\begin{aligned} \sigma'(t) &= \sqrt{2}\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k} \\ \sigma''(t) &= e^t\mathbf{j} + e^{-t}\mathbf{k} \quad . \end{aligned}$$

At $t = 0$ we have

$$\begin{aligned} \sigma(0) &= \left(0, e, e^{-1}\right) \\ \sigma'(t) &= \left(0, e, -e^{-1}\right) \end{aligned}$$

so the equation of the tangent line at $t = 0$ is

$$\mathbf{l}(t) = \sigma(0) + t\sigma'(0) = \left(0, e + et, e^{-1} - e^{-1}t\right)$$

▮

(d) $\sigma(t) = t\mathbf{i} + t\mathbf{j} + \frac{2}{3}t^{3/2}\mathbf{k}$, $t = 9$

- We have

$$\begin{aligned} \sigma'(t) &= \left(1, 1, t^{1/2}\right) \\ \sigma''(t) &= \left(0, 0, \frac{1}{2}t^{-1/2}\right) \quad . \end{aligned}$$

At $t = 9$ we have

$$\begin{aligned} \sigma(9) &= (9, 9, 18) \\ \sigma'(t) &= (1, 1, 3) \end{aligned}$$

so the equation of the tangent line at $t = 9$ is

$$\mathbf{l}(t) = \sigma(9) + t\sigma'(9) = (9 + t, 9 + t, 18 + 3t) \quad .$$

▮

4. Find the path σ such that $\sigma(0) = (0, -5, 1)$ and $\sigma'(t) = \left(t, e^t, t^2\right)$.

- Let $\sigma(t) = (\sigma_x(t), \sigma_y(t), \sigma_z(t))$. We must have

$$\begin{aligned}\sigma_x'(t) &= t\\ \sigma_y'(t) &= e^t\\ \sigma_z'(t) &= t^2\end{aligned}$$

Integrating both sides of these equations with respect to t we obtain

$$\begin{aligned}\sigma_x(t) &= \frac{1}{2}t^2 + A\\ \sigma_y(t) &= e^t + B\\ \sigma_z(t) &= \frac{1}{3}t^3 + C\end{aligned}$$

Here A, B, C are arbitrary constants of integration. A, B, C will be determined by the initial conditions

$$\begin{aligned}0 &= \sigma_x(0) = 0 + A\\ -5 &= \sigma_y(0) = 1 + B\\ 1 &= \sigma_z(0) = C\end{aligned} \qquad \Rightarrow \qquad \begin{aligned}A &= 0\\ B &= -6\\ C &= 1\end{aligned}$$

Thus,

$$\sigma(t) = \left(\frac{1}{2}t^2, e^t - 6, \frac{1}{3}t^3 + 1\right)$$

∎

5. Suppose a particle follows a path $\sigma(t) = (e^t, e^{-t}, \cos(t))$ until it flies off on a tangent at time $t = 1$. Where is it at time $t = 2$.

- At time $t = 1$, the particle is at position

$$\sigma(1) = \left(e, e^{-1}, \cos(1)\right)$$

with velocity

$$\sigma'(1) = \left(e^t, -e^{-t}, -\sin(t)\right)\Big|_{t=1} = \left(e, -e^{-1}, -\sin(1)\right) \quad .$$

The text intends for us to assume that the particle thereafter travels in straight line with velocity $\sigma'(1)$. Thus,

$$\mathbf{r}(t) = \sigma(1) + \sigma'(1)(t-1) = \left(e + e(t-1), e^{-1} - e^{-1}(t-1), \cos(1) - \sin(1)(t-1)\right) \quad .$$

At time $t = 2$ then

$$\mathbf{r}(2) = (2e, 0, \cos(1) - \sin(1)) \quad .$$

∎

Section 2.5

1. Write out the chain rule for each of the following functions and justify your answer in case using Theorem 11.

(a) $\frac{\partial h}{\partial x}$ where $h(x,y) = f(x, u(x,y))$.

- – To compute $\frac{\partial h}{\partial x}$ we regard $h(x,y)$ as a composition of a function $f : \mathbb{R}^2 \to \mathbb{R} : (v,u) \to f(v,u)$ aand a map $g : \mathbb{R}^2 \to \mathbb{R}^2; (x,y) \mapsto (x, u(x,y))$. Then

$$\begin{aligned}Df &= \begin{pmatrix} \frac{\partial f}{\partial v} & \frac{\partial f}{\partial u}\end{pmatrix}\\ Dg &= \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y}\\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y}\end{pmatrix} = \begin{pmatrix} 1 & 0\\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\end{pmatrix}\end{aligned}$$

and so

$$\begin{aligned}\left(\ \tfrac{\partial h}{\partial x}\quad,\quad \tfrac{\partial h}{\partial y}\ \right) = Dh &= D(f\circ g)\\ &= DfDg\\ &= \left(\ \tfrac{\partial f}{\partial v}\quad \tfrac{\partial f}{\partial u}\ \right)\Big|_{(x,u)} * \begin{pmatrix} 1 & 0\\ \tfrac{\partial u}{\partial x} & \tfrac{\partial u}{\partial y}\end{pmatrix}\\ &= \left(\ \tfrac{\partial f}{\partial x}+\tfrac{\partial f}{\partial u}\tfrac{\partial u}{\partial x}\quad,\quad \tfrac{\partial f}{\partial u}\tfrac{\partial u}{\partial y}\ \right)\end{aligned}$$

Therefore, equating the first components on the extreme sides of this equation:

$$\frac{\partial h}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}\quad .$$

$\blacksquare$

(b) $\frac{dh}{dx}$ where $h(x)=f\left(x,u(x),v(x)\right)$.

• In this case, we regard $h(x)$ as a composition of two functions $f:\mathbb{R}^3\ \rightarrow\ \mathbb{R}$, $(w,u,v)\ \mapsto\ f(w,u,v)$ and $g:\mathbb{R}\ \rightarrow\ \mathbb{R}^3$, $x\ \mapsto\ (x,u(x),v(x))$. The general chain rule then gives

$$\begin{aligned}\frac{dh}{dx} &= D\left(f\circ g\right)\\ &= Df\left|_{g(x)}\cdot Dg\right|_x\\ &= \left(\frac{\partial f}{\partial w}\quad \frac{\partial f}{\partial u}\quad \frac{\partial f}{\partial v}\right)\cdot\begin{pmatrix}\frac{dg_1}{ds}\\ \frac{dg_2}{dx}\\ \frac{dg_3}{dx}\end{pmatrix}\\ &= \left(\frac{\partial f}{\partial w}\right)\frac{dg_1}{dx}+\left(\frac{\partial f}{\partial u}\right)\frac{dg_2}{dx}+\left(\frac{\partial f}{\partial v}\right)\frac{dg_3}{dx}\\ &= \left(\frac{\partial f}{\partial x}\right)+\left(\frac{\partial f}{\partial u}\right)\frac{du}{dx}+\left(\frac{\partial f}{\partial v}\right)\frac{dv}{dx}\end{aligned}$$

In the last step, we have simply replaced w by x, g_1 by x, g_2 by $u(x)$, and g_3 by $v(x)$ (in accordance with the definitions of f and g). $\blacksquare$

(c) $\frac{\partial h}{\partial x}$ where $h(x,y,z)=f\left(u(x,y,z),v(x,y),w(x)\right)$.

• In this case, we regard $h(x,y,z)$ as the composition $f\circ g$ of two functions $g:\mathbb{R}^3\ \rightarrow\ \mathbb{R}^3$, $(x,y,z)\ \mapsto$ $(u(x,y,z),v(x,y),w(x))$ and $f:\mathbb{R}^3\ \rightarrow\ \mathbb{R}$. The general chain rule gives

$$\begin{aligned}\left(\frac{\partial h}{\partial x}\quad \frac{\partial h}{\partial y}\frac{\partial h}{\partial z}\right)=Dh &= Df\left|_{g(x,y,z)}\cdot Dg\right|_{(x,y,z)}\\ &= \left(\frac{\partial f}{\partial u},\frac{\partial f}{\partial v},\frac{\partial f}{\partial w}\right)\Bigg|_{g(x,y,z)}\begin{pmatrix}\frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z}\\ \frac{\partial g_3}{\partial x} & \frac{\partial g_3}{\partial y} & \frac{\partial g_3}{\partial z}\end{pmatrix}\end{aligned}$$

So

$$\begin{aligned}\frac{\partial h}{\partial x} &= \left(\frac{\partial f}{\partial u}\right)\left(\frac{\partial g_1}{\partial x}\right)+\left(\frac{\partial f}{\partial v}\right)\left(\frac{\partial g_2}{\partial x}\right)+\left(\frac{\partial f}{\partial w}\right)\left(\frac{\partial g_3}{\partial x}\right)\\ &= \left(\frac{\partial f}{\partial u}\right)\left(\frac{\partial u}{\partial x}\right)+\left(\frac{\partial v}{\partial x}\right)\left(\frac{\partial v}{\partial x}\right)+\left(\frac{\partial f}{\partial w}\right)\left(\frac{dw}{dx}\right)\end{aligned}$$

$\blacksquare$

2. Verify the first special case of the chain rule for the composition $f\circ\mathbf{c}$ in each of the following cases.

(a) $f(x,y)=xy$, $\mathbf{c}(t)=(e^t,\cos(t))$.

• If we compute $f(t)=f\left(\mathbf{c}(t)\right)$ explicitly, we find

$$f\left(\mathbf{c}(t)\right)=x(t)y(t)=e^t\cos(t)\quad ,$$

so

$$\frac{df}{dt} = e^t \cos(t) - e^t \sin(t) \quad .$$

On the other hand, according to the chain rule

$$\begin{aligned}
\frac{df}{dt} = Df &= Df\,\big|_{\mathbf{c}(t)} \cdot D\mathbf{c}\big|_t \\
&= \left(\frac{\partial f}{\partial x} \ , \quad \frac{\partial f}{\partial x} \right)\Bigg|_{\mathbf{c}(t)} \cdot \\
&\ \frac{dc_2}{dt} \\
&= (y(t), x(t)) \cdot \\
&\ -\sin(t) \\
&= \big(\cos(t), e^t\big) \cdot \\
&\ -\sin(t) \\
&= \cos(t)e^t - e^t \sin(t)
\end{aligned}$$

which, of course agrees with the preceding computation. ∎

(b) $f(x,y) = e^{xy} \quad , \quad \mathbf{c}(t) = \left(3t^2, t^3\right) \quad .$

- If we compute $f(t) = f\left(\mathbf{c}(t)\right)$ explicitly, we find

$$f\left(\mathbf{c}(t)\right) = e^{x(t)y(t)} = e^{3t^5} \quad ,$$

so

$$\frac{df}{dt} = 15t^4 e^{3t^5} \quad .$$

On the other hand, according to the chain rule

$$\begin{aligned}
\frac{df}{dt} = Df &= Df\,\big|_{\mathbf{c}(t)} \cdot D\mathbf{c}\big|_t \\
&= \left(\frac{\partial f}{\partial x} \ , \quad \frac{\partial f}{\partial x} \right)\Bigg|_{\mathbf{c}(t)} \cdot \\
&\ \frac{dc_2}{dt} \\
&= \left(y(t)e^{x(t)y(t)}, x(t)e^{x(t)y(t)} \right) \cdot \\
&\ 3t^2 \\
&= \left(t^3 e^{3t^5}, 3t^2 e^{3t^5} \right) \cdot \\
&\ 3t^2 \\
&= 6t^4 e^{3t^5} + 9t^4 e^{3t^5} \\
&= 15t^4 e^{3t^5}
\end{aligned}$$

which, of course agrees with the preceding computation. ∎

3. Let $f : \mathbb{R}^3 \ \rightarrow \ \mathbb{R}$ be differentiable. Making the substitution

$$x = \rho\cos(\theta)\sin(\phi) \quad , \quad y = \rho\sin(\theta)\sin(\phi) \quad , \quad z = \rho\cos(\phi)$$

(spherical coordinates) into $f(x, y, z)$, compute $\partial f/\partial\rho$, $\partial f/\partial\theta$, and $\partial f/\partial\phi$.

- Let $g : \mathbb{R}^3 \ \rightarrow \ \mathbb{R}^3$ be the map sending

$$(\rho, \theta, \phi) \ \rightarrow \ (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi))$$

If we set

$$f\left(\rho, \theta, \phi\right) = f\left(x(\rho,\theta,\phi), y(\rho,\theta,\phi), z(\rho,\theta,\phi)\right)$$

then the Chain Rule tells us that

$$
\begin{aligned}
\left(\tfrac{\partial f}{\partial \rho}\,\tfrac{\partial f}{\partial \theta}\,\tfrac{\partial f}{\partial \phi}\right)\Big|_{(\rho,\theta,\phi)} &= Df\left|_{(x(\rho,\theta,\phi),y(\rho,\theta,\phi),z(\rho,\theta,\phi))}Dg\right|_{(\rho,\theta,\phi)} \\
&= \left(\frac{\partial f}{\partial x}\quad \frac{\partial f}{\partial y}\quad \frac{\partial f}{\partial z}\right)\begin{pmatrix} \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix} \\
&= \left(\frac{\partial f}{\partial x}\quad \frac{\partial f}{\partial y}\quad \frac{\partial f}{\partial z}\right)\begin{pmatrix} \sin(\theta)\sin(\phi) & \rho\cos(\theta)\sin(\phi) & \rho\sin(\theta)\cos(\phi) \\ \cos(\phi) & 0 & -\rho\sin(\phi) \end{pmatrix}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{\partial f}{\partial \rho} &= \frac{\partial f}{\partial x}\cos(\theta)\sin(\phi) + \frac{\partial f}{\partial y}\sin(\theta)\sin(\phi) + \frac{\partial f}{\partial z}\cos(\phi) \\
\frac{\partial f}{\partial \theta} &= -\frac{\partial f}{\partial x}\rho\sin(\theta)\sin(\phi) + \frac{\partial f}{\partial y}\rho\cos(\theta)\sin(\phi) \\
\frac{\partial f}{\partial \phi} &= \frac{\partial f}{\partial x}\rho\cos(\theta)\cos(\phi) + \frac{\partial f}{\partial y}\rho\sin(\theta)\cos(\phi) - \frac{\partial f}{\partial z}\rho\sin(\phi)
\end{aligned}
$$

where, of course, the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$ are all to be evaluated at the point $(x,y,z) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi))$. ∎

4. Let $f(u,v) = \left(\tan(u-1) - e^v, u^2 - v^2\right)$ and $g(x,y) = (e^{x-y}, x-y)$. Calculate $f \circ g$ and $D(f\circ g)(1,1)$.
   - Since both f and g are functions from $\mathbb{R}^2$ to $\mathbb{R}^2$, $f\circ g$ is also a function from $\mathbb{R}^2$ to $\mathbb{R}^2$. We have

$$
\begin{aligned}
(f\circ g)(x,y) &= \left(\tan\left(u(x,y)-1\right) - e^{v(x,y)}, (u(x,y))^2 - (v(x,y))^2\right) \\
&= \left(\tan\left(e^{x-y}-1\right) - e^{x-y}, \left(e^{x-y}\right)^2 - (x-y)^2\right)
\end{aligned}
$$

If we try to compute $D(f\circ g)(1{,}1)$ directly we're going get a mess; we'd have to employ the chain rule for ordinary differentiation a number of times without any simplification appearing until we finally evaluate the result at the point (1,1). Using the chain rule for partial derivatives, we can hope for some simplification as soon as we we evaluate Df at the point $g(1,1) = (1,0)$ and Dg at the point (1,1).

$$
\begin{aligned}
D(f\circ g)|_{(1,1)} &= Df\left|_{g(1,1)}Dg\right|_{(1,1)} \\
&= \left.\begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{pmatrix}\right|_{(u,v)=g(1,1)} \left.\begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{pmatrix}\right|_{(1,1)} \\
& \left.\begin{pmatrix} \sec^2(u-1) & -e^v \\ 2u & -2v \end{pmatrix}\right|_{(u,v)=(1,0)} \left.\begin{pmatrix} e^{x-y} & -e^{x-y} \\ 1 & -1 \end{pmatrix}\right|_{(1,1)} \\
&= \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 1-1 & -1+1 \\ 2+0 & -2+0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 2 & -2 \end{pmatrix}
\end{aligned}
$$

∎

Section 2.6

1. Show that the directional derivative of $f(x,y,z) = z^2x + y^3$ at (1,1,2) in the direction $\left(1/\sqrt{5}, 2/\sqrt{5}, 0\right)$ is $2\sqrt{5}$.
   - According to Theorem 12 on page 147,

$$
D_{\mathbf{n}}f\left|_{(x_o,y_o,z_o)}\right. = \nabla f\left|_{(x_o,y_o,z_o)}\right. \cdot \mathbf{n}\ .
$$

Thus,

$$
\begin{aligned}
D_{\left(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}},0\right)}\left[z^2x+y^3\right]\Big|_{(1,1,2)} &= \left(z^2,3y^2,2zx\right)\big|_{(1,1,2)}\cdot\left(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}},0\right) \\
&= (4,3,4)\cdot\left(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}},0\right) \\
&= \frac{1}{\sqrt{5}}(4+6+0) \\
&= \frac{2\cdot 5}{\sqrt{5}} \\
&= 2\sqrt{5}
\end{aligned}
$$

▮

2. Find the equation of the plane tangent to the surface $z=f(x,y)$ at the indicated point.
   (a) $z=x^3+y^3-6xy$, (1,2,-3).
   - This can be done two different ways.
     The first is to recognize that

$$z=x^3+y^3-6xy$$

is the graph of the function $f(x,y)$. According to the discussion in Section 2.3 (see pages 122 - 123), the equation of the plane tangent to the graph of f at (x_o,y_o) is

$$z=f\left(x_o,y_o\right)+\frac{\partial f}{\partial x}\Big|_{(x_o,y_o)}\left(x-x_o\right)+\frac{\partial f}{\partial y}\Big|_{(x_o,y_o)}\left(y-y_o\right) \quad .$$

For the case at hand, this would lead to the following equation

$$
\begin{aligned}
z &= -3+\left(3x^2-6y\right)\big|_{(1,2)}(x-1)+\left(3y^2-6x\right)\big|_{(1,2)}(y-2) \\
&= -3-9(x-1)+(6)(y-2) \\
&= -9x+6y-6
\end{aligned}
$$

or

$$9x-6y+z=-6 \quad .$$

The second method is to rewrite

$$z=x^3+y^3-6xy$$

as

$$x^3+y^3-6xy-z=0$$

as the equation of a level surface of a function

$$g(x,y,z)=x^3+y^3-6xy-z \quad .$$

According to the definition of page 150, the equation of the tangent plane to a level surface at a point (x_o,y_o,z_o) is

$$\nabla f\left(x_o,y_o,z_o\right)\cdot\left(x-x_o,y-y_o,z-z_o\right)=0 \quad .$$

For the case at hand, we would have

$$
\begin{aligned}
0 &= \left(3x^2-6y,3y^2-6x,-1\right)\big|_{(1,2,-3)}\cdot(x-1,y-2,z+3) \\
&= (-9,6,-1)\cdot(x-1,y-2,z+3) \\
&= -9x+9+6y-12-z-3 \\
&= -9x+6y-z-6
\end{aligned}
$$

or

$$9x-6y+z=-6$$

which, of course, is identical to the result of the first method. ▮

(b) $z = \cos(x)\cos(y)$, $(0, \pi/2, 0)$.

- To find the equation of the tangent plane, we'll use the second method of part (a). The surface $z = \cos(x)\cos(y)$ coincides with the level surface $f(x, y, z) = 0$ of the function

$$f(x, y, z) = \cos(x)\cos(y) - z \quad .$$

Thus, the equation of the tangent plane to the level surface $f(x, y, z) = 0$ at the point $(0, \pi/2, 0)$ is

$$\begin{aligned} 0 &= \nabla f\left(0, \frac{\pi}{2}, 0\right) \cdot \left(x - 0, y - \frac{\pi}{2}, z - 0\right) \\ &= (-\sin(x)\cos(y), -\cos(x)\sin(y), -1)\Big|_{(0,\frac{\pi}{2},0)} \cdot \left(x, y - \frac{\pi}{2}, z\right) \\ &= (0, -1, -1) \cdot \left(x, y - \frac{\pi}{2}, z\right) \\ &= -y + \frac{\pi}{2} - z \end{aligned}$$

or

$$y + z = \frac{\pi}{2} \quad .$$

∎

3. Compute the gradient ∇f for each of the following functions.

(a) $f(x, y, z) = 1/\sqrt{x^2 + y^2 + z^2}$

-

$$\begin{aligned} \nabla f &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\ &= \left(\frac{-x}{[x^2 + y^2 + z^2]^{3/2}}, \frac{-y}{[x^2 + y^2 + z^2]]]]^{3/2}}, \frac{-z}{[x^2 + y^2 + z^2]^{3/2}}\right) \end{aligned}$$

∎

(b) $f(x, y, z) = xy + yz + xz$

-

$$\begin{aligned} \nabla f &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\ &= (y + z, x + z, x + y) \end{aligned}$$

∎

(c) $f(x, y, z) = \frac{1}{x^2+y^2+z^2}$.

-

$$\begin{aligned} \nabla f &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\ &= \left(\frac{-2x}{[x^2 + y^2 + z^2]^{2}}, \frac{-2y}{[x^2 + y^2 + z^2]]]]^{2}}, \frac{-2z}{[x^2 + y^2 + z^2]^{2}}\right) \end{aligned}$$

∎

4. For each of the functions in Exercise 6, what is the direction of fastest increase at (1,1,1)?

- The direction of fastest increase of a differentiable function f at a point (x_o, y_o, z_o) is given by the value of ∇f at that point. Thus, for the three functions in Problem 2.6.6 we have, respectively,

$$\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$$

$$(2, 2, 2)$$

$$\left(\frac{-2}{\sqrt{3}}, \frac{-2}{\sqrt{3}}, \frac{-2}{\sqrt{3}}\right)$$

as the (un-normalized) directions of fastest increase. ∎

5. Captain Ralph is in trouble near the sunny side of Mercury. The temperature of the ship's hull when he is at location (x,y,z) will be given by

$$T(x,y,z) = e^{-x^2-2y^2-3z^2}$$

where x, y, z are measured in meters. He is currently at (1,1,1).

(a) In what direction should he proceed in order to decrease his temperature most rapidly?

- The direction of fastest increase in temperature around the point (1,1,1) is given by the value of the gradient ∇T at that point. The direction of the fastest *decrease* in temperature will be the direction exactly opposite $\nabla T(1,1,1)$. Thus, Captain Ralph should head in the direction

$$\begin{aligned}
-\nabla T(1,1,1) &= -\left(-2xe^{-x^2-2y^2-3z^2}, -4ye^{-x^2-2y^2-3z^2}, -6ze^{-x^2-2y^2-3z^2}\right)\Big|_{(1,1,1)} \\
&= \left(2e^{-6}, 4e^{-6}, 6e^{-6}\right)
\end{aligned}$$

This vector, however, has a non-zero magnitude. To identify the normalized direction vector corresponding to $-\nabla T(1,1,1)$ we simply divide $-\nabla T(1,1,1)$ by its magnitude. This yields

$$\mathbf{n} = \frac{-\nabla T(1,1,1)}{\|-\nabla T(1,1,1)\|} = \frac{\left(2e^{-6}, 4e^{-6}, 6e^{-6}\right)}{\sqrt{56}e^{-6}} = \frac{(1,2,3)}{\sqrt{14}}.$$

∎

(b) If the ship travels at e^8 meters per second, how fast will the temperature be decreasing if he heads in that direction?

- Consider the trajectory given by the map

$$\mathbf{c}(t) = (1,1,1) + \left(e^8 m/s\right)\mathbf{n}t \quad .$$

This would correspond to a straight line trajectory passing through the point (1,1,1) at $t = 0$ in the direction of $\mathbf{n}$ with a speed of $e^8 m/s$. The temperature the ship would feel as it travelled along this trajectory would be

$$T(t) = T(\mathbf{c}(t))$$

and the rate at which the temperature would decrease would be

$$\frac{dT}{dt} = \frac{d}{dt}T(\mathbf{c}(t)) = DT\left|_{\mathbf{c}(t)} \cdot D\mathbf{c}\right|_t = \nabla T(\mathbf{c}(t)) \cdot \frac{d\mathbf{c}}{dt}(t) \quad .$$

At $t = 0$, we have $\mathbf{c}(t) = (1,1,1)$ and $\frac{d\mathbf{c}}{dt} = e^8(m/s)\mathbf{n}$. Thus,

$$\begin{aligned}
\frac{dT}{dt} &= \nabla T(1,1,1) \cdot \left(\frac{e^8}{\sqrt{14}}, \frac{2e^8}{\sqrt{14}}, \frac{3e^8}{\sqrt{14}}\right) \\
&= \left(-2e^{-6}, -4e^{-6}, -6e^{-6}\right) \cdot \left(\frac{e^8}{\sqrt{14}}, \frac{2e^8}{\sqrt{14}}, \frac{3e^8}{\sqrt{14}}\right) \\
&= \frac{e^2}{\sqrt{14}}(-2-8-18) \\
&= -2\sqrt{14}e^2
\end{aligned}$$

∎

(c) Unfortunately, the metal of the hull will crack if cooled at a rate greater than $\sqrt{14}e^2$ degrees per second. Describe the set of possible directions in which he may proceed to bring the temperature down at no more than that rate.

- We now consider straight line trajectories through the point (1,1,1) of the form

$$\mathbf{c}'(t) = (1,1,1) + e^8(m/s)\mathbf{n}'t$$

where now $\mathbf{n}'$ is an arbitrary unit vector (indicating the direction at which the spaceship is heading). Along such a path we have

$$\begin{aligned}
\frac{dT}{dt}(0) &= \left.\frac{d}{dt}T\left(\mathbf{c}'(t)\right)\right|_{t=0} \\
&= \nabla T\Big|_{\mathbf{c}'(0)} \cdot \left.\frac{d\mathbf{c}'}{dt}\right|_0 \\
&= \nabla T(1,1,1)\cdot e^8\mathbf{n}'
\end{aligned}$$

Using the fact that

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\|\,\cos(\theta)$$

we can write

$$\begin{aligned}
\frac{dT}{dt}(0) &= \|\nabla T(1,1,1)\|\,\|e^8\mathbf{n}'\|\,\cos(\theta) \\
&= \left(\sqrt{56}e^{-6}\right)\left(e^8\right)\cos(\theta) \\
&= 2\sqrt{14}e^2\cos(\theta)
\end{aligned}$$

We now demand the rate at which the spaceship hull cools be no greater than $\sqrt{14}e^2$;

$$-\sqrt{14}e^2 < \frac{dT}{dt}(0) = 2\sqrt{14}e^2\cos(\theta)$$

which leads to the condition that

$$\cos(\theta) > -\frac{1}{2}$$

or

$$\theta < 120^\circ$$

We conclude that the angle between the direction of fastest increase in temperature and the direction in which the spaceship heads should be no greater than 120°. ∎

6. Compute the second partial derivatives $\partial^2 f/\partial x^2$, $\partial^2 f/\partial x\partial y$, $\partial^2 f/\partial y\partial x$, $\partial^2 f/\partial y^2$ for each of the following functions. Verify Theorem 15 in each case.
   (a) $f(x,y) = 2xy/\left(x^2+y^2\right)^2$, $(x,y)\neq 0$.

•

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{2y\left(x^2+y^2\right)^2 - (2xy)2\left(x^2+y^2\right)(2x)}{\left(x^2+y^2\right)^4} \\
&= \frac{2y^3-6x^2y}{\left(x^2+y^2\right)^3}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{2x\left(x^2+y^2\right)^2 - (2xy)2\left(x^2+y^2\right)(2y)}{\left(x^2+y^2\right)^4} \\
&= \frac{2x^3-6xy^2}{\left(x^2+y^2\right)^3}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial y}\frac{\partial f}{\partial x} &= \frac{\left(6y^2-6x^2\right)\left(x^2+y^2\right)^3 - \left(2y^3-6x^2y\right)3\left(x^2+y^2\right)^2(2y)}{\left(x^2+y^2\right)^6} \\
&= \frac{-6x^4-6y^4+36x^2y^2}{\left(x^2+y^2\right)^4}
\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial x}\frac{\partial f}{\partial y} &= \frac{\left(6x^2-6y^2\right)\left(x^2+y^2\right)^3-\left(2x^3-6xy^2\right)3\left(x^2+y^2\right)^2(2y)}{\left(x^2+y^2\right)^6} \\ &= \frac{-6x^4-6y^4+36x^2y^2}{\left(x^2+y^2\right)^4}\end{aligned}$$

Note that

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\frac{\partial f}{\partial y} \quad .$$

▮

(b) $f(x,y,z)=e^z+(1/x)+xe^{-y}$, $x\neq 0$.

•

$$\begin{aligned}\frac{\partial f}{\partial x} &= -\frac{1}{x^2}+e^{-y} \\ \frac{\partial}{\partial y}\frac{\partial f}{\partial x} &= -e^{-y} \\ \frac{\partial}{\partial z}\frac{\partial f}{\partial x} &= 0\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y} &= -xe^{-y} \\ \frac{\partial}{\partial x}\frac{\partial f}{\partial y} &= -e^{-y} \\ \frac{\partial}{\partial z}\frac{\partial f}{\partial y} &= 0\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial z} &= e^z \\ \frac{\partial}{\partial x}\frac{\partial f}{\partial z} &= 0 \\ \frac{\partial}{\partial y}\frac{\partial f}{\partial z} &= 0\end{aligned}$$

Note that

$$\begin{aligned}\frac{\partial}{\partial x}\frac{\partial f}{\partial y} &= -e^{-y}=\frac{\partial}{\partial y}\frac{\partial f}{\partial x} \quad , \\ \frac{\partial}{\partial x}\frac{\partial f}{\partial z} &= 0=\frac{\partial}{\partial z}\frac{\partial f}{\partial x} \\ \frac{\partial}{\partial y}\frac{\partial f}{\partial z} &= 0=\frac{\partial}{\partial z}\frac{\partial f}{\partial y}\end{aligned}$$

▮

7. Let

$$f(x,y)=\begin{cases} xy\left(x^2-y^2\right)/\left(x^2+y^2\right) & , \quad (x,y)\neq(0,0) \\ 0 & , \quad (x,y)=0\end{cases}$$

(a) If $(x,y)\neq 0$, calculate $\partial f/\partial x$ and $\partial f/\partial y$.

- For all points $(x,y)\neq 0$, we have

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\left(3x^2y-y^3\right)\left(x^2+y^2\right)-\left(x^3y-xy^3\right)(2x)}{\left(x^2+y^2\right)^2} \\ &= \frac{x^4y+4x^2y^3-y^5}{\left(x^2+y^2\right)^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\left(x^3-3xy^2\right)\left(x^2+y^2\right)-\left(x^3y-xy^3\right)(2y)}{\left(x^2+y^2\right)^2} \\ &= \frac{x^5-4x^3y^2-xy^4}{\left(x^2+y^2\right)^2}\end{aligned}$$

▮

(b) Show that

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = 0 = \left.\frac{\partial f}{\partial y}\right|_{(0,0)}$$

•

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{(x,y)\to(0,0)}$$

At the point (0,0) we will have to be more careful, because $f(x,y)$ is not obviously differentiable there.

$$\begin{aligned}\left.\frac{\partial f}{\partial x}\right|_{(0,0)} &\equiv \lim_{h\to 0}\tfrac{f(h,0)-f(0,0)}{h} \\ &= \lim_{h\to 0}\frac{0-0}{h} \\ &= 0\end{aligned}$$

$$\begin{aligned}\left.\frac{\partial f}{\partial y}\right|_{(0,0)} &\equiv \lim_{h\to 0}\tfrac{f(0,h,)-f(0,0)}{h} \\ &= \lim_{h\to 0}\frac{0-0}{h} \\ &= 0\end{aligned}$$

▮

(c) Show that

$$\left.\frac{\partial^2 f}{\partial x\partial y}\right|_{(0,0)} = 1 \quad , \quad \left.\frac{\partial^2 f}{\partial y\partial x}\right|_{(0,0)} = -1 \quad .$$

- Again, we have to be a little careful evaluating the partial derivatives at the point (0,0). We have

$$\begin{aligned}\left.\frac{\partial}{\partial y}\frac{\partial f}{\partial x}\right|_{(0,0)} &= \lim_{h\to 0}\frac{\left.\frac{\partial f}{\partial x}\right|_{(0,h)} - \left.\frac{\partial f}{\partial x}\right|_{(0,0)}}{h} \\ &= \lim_{h\to 0}\frac{-h-0}{h} \\ &= -1\end{aligned}$$

$$\begin{aligned}\left.\frac{\partial}{\partial x}\frac{\partial f}{\partial y}\right|_{(0,0)} &= \lim_{h\to 0}\frac{\left.\frac{\partial f}{\partial y}\right|_{(h,0)} - \left.\frac{\partial f}{\partial y}\right|_{(0,0)}}{h} \\ &= \lim_{h\to 0}\frac{h-0}{h} \\ &= 1\end{aligned}$$

(d) What went wrong? Why are the mixed partials not equal?
The second partial derivatives exist, but they are not continuous as functions of two variables. Therefore, Theorem 15 can not be applied in this case. ▮

8. A function $u=f(x,y)$ with continuous second partial derivatives satisfying Laplace's equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

is called a *harmonic function*. Show that $u(x,y) = x^3 - 3xy^2$ is harmonic.

•

$$\begin{aligned} \frac{\partial u}{\partial x} &= 3x^2 - 3y^2 \\ \frac{\partial^2 u}{\partial x^2} &= 6x \\ \frac{\partial u}{\partial y} &= -6xy \\ \frac{\partial^2 u}{\partial y^2} &= -6x \end{aligned}$$

Therefore

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0 \quad ,$$

and so u is harmonic. ∎