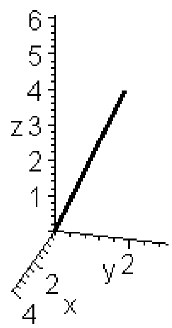
Math 4013

Solutions to Homework Problems for Chapter 1

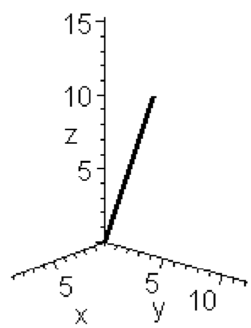
1. Let $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and $\mathbf{w} = \mathbf{i} - \mathbf{j} + \mathbf{k}$. Compute $\mathbf{v} + \mathbf{w}$, $\mathbf{3v}$, $6\mathbf{v} + 8\mathbf{w}$, $-2\mathbf{v}$, $\mathbf{v} \cdot \mathbf{w}$, $\mathbf{v} \times \mathbf{w}$. Interprete each geometrically by graphing the vectors.

$$\mathbf{v} = (3, 4, 5)$$
$$\mathbf{w} = (1, -1, 1)$$

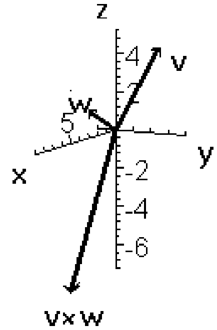
- $\mathbf{v} + \mathbf{w} = (3 + 1, 4 + (-1), 5 + 1) = (4, 3, 6)$



- $3\mathbf{v} = (9, 12, 15)$



- $6\mathbf{v} + 8\mathbf{w} = (18, 24, 30) + (8, -8, 8) = (26, 16, 38)$
- $-2\mathbf{v} = (-6, -8, -10)$
- $\mathbf{v} \cdot \mathbf{w} = (3)(1) + (4)(-1) + (5)(1) = 3 - 4 + 5 = 4$
- $\mathbf{v} \times \mathbf{w} = ((4)(1) - (5)(-1), (5)(1) - (3)(1), (3)(-1) - (4)(1)) = (9, 2, -7)$



$\square$

2.
(a) Find the equation of the line through (-1,2,-1) in the direction of **j**.

- An Equation of a Line has the form

$$\mathbf{r}(t) = \mathbf{r}_o + t\mathbf{v}$$

where $\mathbf{r}_o$ is the position vector of a point on the line and $\mathbf{v}$ is a vector representing the direction of the line. We thus have, for the case at hand,

$$\mathbf{r}(t) = (-1, 2, -1) + t\,(0, 1, 0) = (-1, 2 + t, -1) \quad .$$

$\square$

(b) Find the equation of the line passing through (0,2,-1) and (-3,1,0).

- The line through $\mathbf{a} = (0, 2, -1)$ and $\mathbf{b} = (-3, 1, 0)$, definitely passes through the point $\mathbf{a}$ and has the same direction as $\mathbf{b} - \mathbf{a}$. Thus, its equation is

$$\mathbf{r}(t) = \mathbf{a} + t\,(\mathbf{b} - \mathbf{a}) = (0, 2, -1) + t\,(-3, -1, 1) = (-3t, 2 - t, -1 + t) \quad .$$

$\square$

(c) Find the equation for the plane perpendicular to (-2,1,2) and passing through (-1,1,3).

- Let $\mathbf{r} = (x, y, z)$ be a point in the plane. Then the line through $\mathbf{r}$ and $\mathbf{r}_o = (1, 1, 1)$ must lie entirely in the plane; and so must be perpendicular to the normal vector to the plane, $\mathbf{n} = (-1, 1, -1)$. This leads to the condition

$$\begin{aligned} 0 &= \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_o) \\ &= (-1, 1, -1) \cdot (x - 1, y - 1, z - 1) \\ &= -x + 1 + y - 1 - z + 1 \\ &= -x + y - z + 1 \end{aligned}$$

or

$$x - y + z = 1 \quad .$$

The last equation is the equation of the plane. $\square$

3. Compute $\mathbf{v} \cdot \mathbf{w}$ for the following sets of vectors.

(a) $\mathbf{v} = -\mathbf{i} + \mathbf{j} = (-1, 1, 0); \quad \mathbf{w} = \mathbf{k} = (0, 0, 1)$.

(b) $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} = (1, 2, -1); \quad \mathbf{w} = 3\mathbf{i} + \mathbf{j} = (3, 1, 0)$.

(c) $\mathbf{v} = -2\mathbf{i} - \mathbf{j} + \mathbf{k} = (-2, -1, 1); \quad \mathbf{w} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} = (3, 2, -2)$.

- Using the formula

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

we find
(a)

$$\mathbf{v} \cdot \mathbf{w} = (-1)(0) + (1)(0) + (0)(1) = 0$$

(b)

$$\mathbf{v} \cdot \mathbf{w} = (-1)(3) + (2)(1) + (-1)(0) = -1$$

(c)

$$\mathbf{v} \cdot \mathbf{w} = (-2)(3) + (-1)(2) + (1)(-2) = -10$$

$\square$

4. Compute $\mathbf{v} \times \mathbf{w}$ for the vectors in Exercise 3.

- Using the formula

$$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1)$$

we find
(a)

$$\mathbf{v} \times \mathbf{w} = ((1)(1) - (0)(0), (0)(0) - (-1)(1), (-1)(0) - (1)(0)) = (1, 1, 0))$$

(b)

$$\mathbf{v} \times \mathbf{w} = ((2)(0) - (-1)(1), (-1)(3) - (1)(0), (1)(1) - (2)(3)) = (1, -3, -5)$$

(c)

$$\mathbf{v} \times \mathbf{w} = ((-1)(-2) - (1)(3), (1)(3) - (-2)(-2), (-2)(2) - (-1)(3)) = (-1, -1, -1)$$

$\square$

5. Convert the point (0,3,4) from Cartesian to cylindrical and spherical coordinates.

- The relationship between Cartesian and cylindrical coordinates is given by

$$\begin{array}{rclcrcl} x &=& r\cos\theta & \quad & r &=& \sqrt{x^2+y^2} \\ y &=& r\sin\theta & & \theta &=& \tan^{-1}\left(\frac{y}{x}\right) \\ z &=& z & & z &=& z \end{array}$$

and the relationship between Cartesian and spherical coordinates is given by

$$\begin{array}{rclcrcl} x &=& \rho\cos\theta\sin\phi & \quad & \rho &=& \sqrt{x^2+y^2+z^2} \\ y &=& \rho\sin\theta\sin\phi & & \theta &=& \tan^{-1}\left(\frac{y}{x}\right) \\ z &=& \rho\cos\phi & & \phi &=& \cos^{-1}\left(\frac{z}{\rho}\right) \end{array}$$

We thus have

$$\begin{aligned} r &= \sqrt{0^2+3^2} = 3 \\ \theta &= \tan^{-1}\left(\frac{3}{0}\right) = \tan^{-1}(+\infty) = \frac{\pi}{2} \\ z &= 4 \end{aligned}$$

for the polar coordinates of the point (0,3,4) and

$$\begin{aligned} \rho &= \sqrt{0^2+3^2+4^2} = 5 \\ \theta &= \tan^{-1}\left(\frac{3}{0}\right) = \tan^{-1}(+\infty) = \frac{\pi}{2} \\ \phi &= \cos^{-1}\left(\frac{4}{5}\right) = \end{aligned}$$

are the spherical coordinates of this point. $\square$

6. Convert the point $(1, \pi/4, 1)$ from cylindrical to Cartesian and spherical coordinates.

- Using the formulas appearing in the solution to Problem 22(a) we have We thus have

$$x = r\cos\theta = 1\cdot\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$
$$y = r\sin\theta = 1\cdot\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$
$$z = z = 1$$

for the Cartesian coordinates of the point (0,3,4); and so

$$\rho = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 + (1)^2} = \sqrt{2}$$
$$\theta = \tan^{-1}\left(\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$
$$\phi = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

are the spherical coordinates of this point. □

7. Convert the point $(1, \pi/2, \pi)$ from spherical to Cartesian and cylindrical coordinates.

- The Cartesian coordinates of this point can be obtained from the formulas given in the solution to Problem 22(a). We have and cylindrical coordinates

$$x = \rho\cos\theta\sin\phi = (1)\cos\left(\frac{\pi}{2}\right)\sin(\pi) = 0$$
$$y = \rho\sin\theta\sin\phi = (1)\sin\left(\frac{\pi}{2}\right)\sin(\pi) = 0$$
$$z = \rho\cos\phi = (1)\cos(\pi) = -1$$

Using these Cartesian coordinates we can easily compute the corresponding cylindrical coordinates:

$$r = \sqrt{0^2 + 0^2} = 0$$
$$\theta = \tan^{-1}\left(\frac{0}{0}\right) = \text{undefined}$$
$$z = z = -1$$

Do not be upset by the fact that θ is undefined. In polar coordinates, θ is undefined (read inconsequential) for all points along the z-axis. □