LECTURE 24

The Integral Theorems of Vector Analysis

1. The Fundamental Theorem of Calculus

THEOREM 24.1. *Suppose f is a function that's differentiable on the interval $[a,b]$. Then*

$$\int_a^b \frac{df}{dx}dx = f(b) - f(a)$$

As we saw in Chapter 7, this theorem has the following generalization to line integrals over paths.

THEOREM 24.2. *Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is a function that differentiable at every point along the path $\sigma : [a,b] \to \mathbb{R}^3$. Then*

$$\int_\sigma \nabla f \cdot d\mathbf{s} = f(\sigma(b)) - f(\sigma(a))$$

2. Green's and Stokes'Theorem

THEOREM 24.3. ***(Green's Theorem.)*** *Let $D \subset \mathbb{R}^2$ be any region of the plane that is both Type I and TypeII and let ∂D denote it boundary (oriented counter-clockwise). If $\mathbf{F} = (F_x, F_y, 0)$ is a C^1 vector field on D (regarded now as a surface in $\mathbb{R}^3$), then*

$$\int_D (\nabla \times \mathbf{F})_z \, dA = \int_{\partial D} \mathbf{F} \cdot d\mathbf{s}$$

THEOREM 24.4. ***(Stokes' Theorem.)*** *Let S be an oriented surface defined by a one-to-one parameterization $\Phi : D \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$. Let ∂S denote the oriented boundary of S and let $\mathbf{F}$ be a C_1 vector field on S. Then*

$$\int_S (\nabla \times \mathbf{F})_z \cdot d\mathbf{S} \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

3. Gauss' Theorem

THEOREM 24.5. *Let W be an elementary region in $\mathbb{R}^3$, and let ∂W denote the oriented closed surface that bounds W. Then if $\mathbf{F}$ is any smooth vector field on W*

$$\int_W (\nabla \cdot \mathbf{F}) \, dV = \int_{\partial W} \mathbf{F} \cdot d\mathbf{S}$$

4. The Fundamental Idea

REMARK 24.6. Note that each of these theorems can be thought of as relating the integral of a "derivative" of a function over a region to a sum over the values of the function on the boundary of that region.

5. Application to Maxwell's Equations

Recall the differential form of Maxwell's Equations

$$
\begin{array}{ll}
\nabla \cdot \mathbf{E} = \frac{1}{4\pi\varepsilon_o}\rho(\mathbf{x}) & \text{(Gauss' Law)} \\
\nabla \cdot \mathbf{B} = 0 & \text{(Gauss' Law for Magnetic Field)} \\
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} & \text{(Faraday's Law)} \\
\nabla \times \mathbf{B} = \mu_o\varepsilon_o\frac{\partial \mathbf{E}}{\partial t} + \mu_o\mathbf{j}(\mathbf{x}) & \text{(Ampere's Law)}
\end{array}
$$

If we integrate the differential form of Gauss' Law over a volume W then Gauss' Theorem yields the integral form of Gauss's Law:

$$
\begin{aligned}
\int_{\partial W} \mathbf{E} \cdot d\mathbf{S} &= \int_W (\nabla \cdot \mathbf{E})\, dV \\
&= \int_W \frac{1}{4\pi\varepsilon_o}\rho(\mathbf{x}) dV \\
&= \frac{Q}{4\pi\varepsilon_o}
\end{aligned}
$$

Similarly, integrating the divergence of the magnetic field over a volume W yields

$$
\begin{aligned}
\int_{\partial W} \mathbf{B} \cdot d\mathbf{S} &= \int_W (\nabla \cdot \mathbf{B})\, dV \\
&= \int_{\partial W} 0 dV \\
&= 0
\end{aligned}
$$

which is the integral form of Gauss' Law for Magnetic Fields.

If we integrate the differential form of Faraday's Law over a surface S we obtain from Stokes' Theorem

$$
\begin{aligned}
\int_{\partial S} \mathbf{E} \cdot d\mathbf{s} &= \int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} \\
&= \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \\
&= -\frac{\partial}{\partial t}\int_S \mathbf{B} \cdot d\mathbf{S}
\end{aligned}
$$

which is the integral form of Faraday's Law.

Similarly if we integrate the differential form of Ampere's Law over a surface S we obtain from Stokes' Theorem

$$
\begin{aligned}
\int_{\partial S} \mathbf{B} \cdot d\mathbf{s} &= \int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S} \\
&= \int_S \left(\mu_o\varepsilon_o\frac{\partial \mathbf{E}}{\partial t} + \mu_o\mathbf{j}(\mathbf{x})\right) \cdot d\mathbf{S} \\
&= \mu_o\varepsilon_o\frac{\partial}{\partial t}\int_S \mathbf{E} \cdot d\mathbf{S} + \mu_o I
\end{aligned}
$$

which is the integral form of Faraday's Law.

We thus arrive at the following integral formulation of Maxwell's Equations

$$\int_{\partial W} \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{4\pi\varepsilon_o}$$

$$\int_{\partial W} \mathbf{B} \cdot d\mathbf{S} = 0$$

$$\int_{\partial S} \mathbf{E} \cdot d\mathbf{s} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S}$$

$$\int_{\partial S} \mathbf{B} \cdot d\mathbf{s} = \mu_o \varepsilon_o \frac{\partial}{\partial t} \int_S \mathbf{E} \cdot d\mathbf{S} + \mu_o I$$