LECTURE 23

Integrals over Surfaces

1. Parameterized Surfaces

DEFINITION 23.1. *A **parameterized surface** is a continuous 1:1 map $\Phi : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^n$. The **surface** S corresponding to Φ is the image of the domain D in the target space $\mathbb{R}^n$:*

$$S = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \Phi(u,v) \text{ for some } (u,v) \in D\}$$

If we write

$$\Phi(u,v) = (x_1(u,v), x_2(u,v), \ldots, x_n(u,v)) \in \mathbb{R}^n$$

and the component functions $x_i(u,v)$ are all of class C_1, then we say that S is a surface of class C_1.

EXAMPLE 23.2. Graphs of functions from $f : \mathbb{R}^2 \rightarrow \mathbb{R}$.

Define

$$\Phi(u,v) : (u,v) \rightarrow (u, v, f(u,v))$$

Then Φ will be a parameterized surface. The corresponding surface is just the graph of f.

EXAMPLE 23.3. The Sphere

Take

$$\Phi(u,v) : [0,2\pi] \times [0,\pi] \rightarrow \mathbb{R}^3 \quad : \quad \Phi(u,v) = (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$$

Then the corresponding surface will be a sphere of radius 1 centered about the origin.

2. Tangent Plane to a Surface

Suppose that $\Phi : D \rightarrow \mathbb{R}^3$ is a parameterized surface that is differentiable at the point $(u_o, v_o) \in D$. Keeping v fixed at v_o, we obtain a path

$$\sigma_1 : \mathbb{R} \rightarrow \mathbb{R}^3 \quad : \quad \sigma_1(t) = \Phi(u_o + t, v_o)$$

The tangent vector to this path at the point (u_o, v_o) is just

$$\mathbf{T}_u = \left.\frac{d\sigma_1}{dt}\right|_{t=0} = \left.\left(\frac{\partial \Phi_x}{\partial u} + \frac{\partial \Phi_y}{\partial u} + \frac{\partial \Phi_z}{\partial u}\right)\right|_{(u_o,v_o)}$$

Similarly, we can keep u fixed at u_o and construct a curve be varing v:

$$\sigma_2(t) = \Phi(u_o, v_o + t)$$

The tangent vector to the curve σ_2 at the point (u_o, v_o) will be

$$\mathbf{T}_v = \left.\frac{d\sigma_2}{dt}\right|_{t=0} = \left.\left(\frac{\partial \Phi_x}{\partial v} + \frac{\partial \Phi_y}{\partial v} + \frac{\partial \Phi_z}{\partial v}\right)\right|_{(u_o,v_o)}$$

Now since the paths σ_1 and σ_2 both lie entirely within the surface S, their tangent vectors should also lie within S; or at least lie within the plane tangent to S at the point $\Phi(u_o, v_o)$. Indeed, we can use these tangent vectors to prescribe the plane tangent to S at the point $\Phi(u_o, v_o)$. Set

$$\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v$$

This vector should be perpendicular to every line in the tangent plane. This observation motivates the following definition.

DEFINITION 23.4. *Let $\Phi : D \to \mathbb{R}^3$ be a parameterized surface that is differentiable at the point $(u_o, v_o) \in D$. The plane tangent to the surface $S = \Phi(D)$ at the point $\Phi(u, v)$ is the plane defined by*

$$\mathbf{T}S_{(u_o,v_o)} = \left\{ \mathbf{x} \in \mathbb{R}^3 \mid (\mathbf{x} - \Phi(u_o, v_o)) \cdot (\mathbf{T}_u \times \mathbf{T}_v) = 0 \right\}$$

3. Surface Integrals of Scalar Functions

DEFINITION 23.5. *Let $f(\mathbf{x})$ be a real-valued function on $\mathbb{R}^3$ and let $\Phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parameterized surface. The integral of f over the surface $S = \Phi(D)$ is the integral*

$$\int_S f dS \equiv \int_D f(\Phi(u, v)) \|\mathbf{T}_u \times \mathbf{T}_v\| \, dudv$$

REMARK 23.6. The area of a surface is just the integral

$$\int_S dS \equiv \int_D \|\mathbf{T}_u \times \mathbf{T}_v\| \, dudv$$

EXAMPLE 23.7. Let S be the upper hemisphere of the unit sphere in $\mathbb{R}^3$.

$$S = \left\{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \ , \ z \geq 0 \right\}$$

Calculate

$$\int_S z dS$$

- We can realize this sphere as the image of the following parameterized surface

  $$\Phi : [0, 2\pi] \times [0, \frac{\pi}{2}] \to \mathbb{R}^3 \quad , \quad \Phi(\theta, \phi) = (\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi))$$

  We then have

  $$\begin{aligned} \mathbf{T}_\theta &= (-\sin(\theta)\sin(\phi), \cos(\theta)\sin(\phi), 0) \\ \mathbf{T}_\phi &= (\cos(\theta)\cos(\phi), \sin(\theta)\cos(\phi), -\sin(\phi)) \end{aligned}$$

  and so

  $$\begin{aligned} \mathbf{T}_\theta \times \mathbf{T}_\phi &= \left(-\cos(\theta)\sin^2(\phi) - 0, 0 - \sin(\theta)\sin^2(\phi), -\sin^2(\theta)\sin(\phi)\cos(\phi) - \cos^2(\theta)\sin(\phi)\cos(\phi) \right) \\ &= \left(-\cos(\theta)\sin(\phi), \sin(\theta)\sin^2(\phi), -\sin(\phi)\cos(\phi) \right) \end{aligned}$$

  and

  $$\begin{aligned} \|\mathbf{T}_\theta \times \mathbf{T}_\phi\|^2 &= \cos^2(\theta)\sin^4(\phi) + \sin^2(\theta)\sin^4(\phi) + \sin^2(\phi)\cos^2(\phi) \\ &= \sin^2(\phi)\left(\sin^2(\phi) + \cos^2(\phi)\right) \\ &= \sin^2(\phi) \end{aligned}$$

  $$\Rightarrow \quad \|\mathbf{T}_\theta \times \mathbf{T}_\phi\| = |\sin(\phi)|$$

Hence,

$$\begin{aligned}
\int_S z dS &= \int_0^{2\pi}\int_0^{\pi/2} z(\theta,\phi)\left\|\mathbf{T}_\theta \times \mathbf{T}_\phi\right\| d\phi d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/2} \cos(\phi)\sin(\phi)\, d\phi\, d\theta \\
&= \int_0^{2\pi}\left(\int_0^1 u du\right) d\theta \\
&= \int_0^{2\pi}\frac{1}{2} d\theta \\
&= \pi
\end{aligned}$$

(In the third line we employed the substitution $u = \sin(\phi)$.)

4. Surface Integrals of Vector-Valued Functions

Definition 23.8. *Let* $\mathbf{F}$ *be a vector field on* $\mathbb{R}^3$ *and let* $\Phi : D \to \mathbb{R}^3$ *be a parameterized surface. The surface integral of* $\mathbf{F}$ *over the surface* $S = \Phi(D)$ *is the integral*

$$\int_\Phi \mathbf{F}\cdot d\mathbf{S} \equiv \int_D \mathbf{F}\left(\Phi(u,v)\right)\cdot\left(\mathbf{T}_u \times \mathbf{T}_v\right)\, du\, dv$$

5. Orientable Surfaces

When one computes the work done in moving an object along a path $\sigma : [a,b] \to \mathbb{R}^3$, it is important that σ moves in the correct direction; in particular

$$\begin{aligned}
\sigma(a) &= \text{ the initial point} \\
\sigma(b) &= \text{ the ending point}
\end{aligned}$$

Otherwise, the work integral

$$W = \int_\sigma \mathbf{F}\cdot d\mathbf{s}$$

will yield the negative of the correct result. Thus, the notion of a physical trajectory is more than just a collection of points in space, it also should include a certain orientation indicating which direction the object is moving.

The situation is similar for surfaces; but here the notion of orientation has to do with the ambiguity in the sign of the normal vector

$$\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v = -\mathbf{T}_v \times \mathbf{T}_u$$