LECTURE 22

Integrals over Curves

Recall that the arc length of a parameterized curve $\sigma : [a, b] \to \mathbb{R}^n$ is given by

$$L = \int_a^b \left\| \frac{d\sigma}{dt} \right\| dt$$

This integral should be thought of as the limit of a Riemann sum of the form

$$\sum L_i = \sum_i \left\| \frac{d\sigma}{dt}(t_i) \right\| \Delta t$$

where the L_i is the length of the curve segment between $\sigma(t_i)$ and $\sigma(t_i + \Delta t)$.

It is sometimes useful to consider "weighted" Riemann sums of the form

$$\sum f(\sigma(t_i)) L_i = \sum_i f(\sigma(t_i)) \left\| \frac{d\sigma}{dt}(t_i) \right\| \Delta t$$

In which case we are lead to consider integrals of the form

$$\int_a^b f(\sigma(t)) \left\| \frac{d\sigma}{dt} \right\| dt$$

Such integrals are called **path integrals** and are commonly presented via the notation

$$\int_C f ds$$

where $C = \{\sigma(t) \in \mathbb{R}^n \mid t \in [a, b]\}$ denotes the corresponding curve.

For example, if we wished to calculate the moment of inertia about the y-axis of a wire winding through the xy-plane we might consider a Riemann sum of the form

$$\sum_i x(\sigma(t)) \rho \left\| \frac{d\sigma}{dt}(t_i) \right\| \Delta t$$

Here ρ is the density of the wire, so that

$$\rho \left\| \frac{d\sigma}{dt}(t_i) \right\| \Delta t$$

is interpretable as the mass of the wire lying between $\sigma(t_i)$ and $\sigma(t_i + \Delta t)$; and $x(\sigma(t))$ is the distance of that segment from the y-axis. Passing from the Riemann sum to an integral expression in the usual fashion yields an integral fo the form

$$\int_a^b x(\sigma(t)) \rho \left\| \frac{d\sigma}{dt}(t_i) \right\| dt \equiv \int_C x ds$$

0.1. Line Integrals. Another kind of integral that arises frequently in applications is the so-called **line integral**. This is defined as follows.

DEFINITION 22.1. *Let* $\mathbf{F}$ *be a vector field on* $\mathbb{R}^n$ *and let* $\sigma : [a,b] \to \mathbb{R}^n$ *be a parameterized path in* $\mathbb{R}^n$. *The* ***line integral*** *of* $\mathbf{F}$ *along the corresponding curve* $C = \{\sigma(t) \in \mathbb{R}^n \mid t \in [a,b]\}$ *is the integral*

$$\int_C \mathbf{F} \cdot d\mathbf{s} \equiv \int_a^b \mathbf{F}(\sigma(t)) \cdot \frac{d\sigma}{dt} dt$$

EXAMPLE 22.2. Let $\mathbf{F}(\mathbf{x})$ be a vector field describing the total force acting on a particle at position $\mathbf{x}$. The work done in moving the particle a small displacement $\Delta\mathbf{x}$ is given by

$$\Delta W = \mathbf{F} \cdot \Delta\mathbf{x}$$

If we seek to estimate the work done in moving a particle along a path $\sigma : [a,b] \to \mathbb{R}^n$ we are then led to a Riemann sum of the form

$$W = \sum \Delta W = \sum \mathbf{F}(\mathbf{x}_i) \cdot \Delta\mathbf{x} = \sum \mathbf{F}(\mathbf{x}_i) \cdot \frac{d\sigma}{dt} \Delta t$$

and hence to an integral of the form

$$\int_C \mathbf{F} \cdot d\mathbf{s} \equiv \int_a^b \mathbf{F}(\sigma(t)) \cdot \frac{d\sigma}{dt} dt$$

0.2. Properties of Path Integrals and Line Integrals.

DEFINITION 22.3. *Let* $h(t)$ *be a differentiable real-valued function mapping an interval* $[c,d]$ *on the real line to another interval* $[a,b]$. *Assume moreover that* $h(t)$ *is 1:1 and increasing. Let* $\sigma : [a,b] \to \mathbb{R}^n$ *be a piecewise differentiable path. Then the path*

$$\gamma = \sigma \circ h : [c,d] \to \mathbb{R}^n$$

is called a ***reparameterization*** *of* σ.

THEOREM 22.4. *If* $\gamma : [c,d] \to \mathbb{R}^n$ *is a reparameterization of a path* $\sigma : [a,b] \to \mathbb{R}^n$ *then*

1. *For any function* $f : \mathbb{R}^n \to \mathbb{R}$
$$\int_\sigma f ds = \int_\gamma f ds$$
2. *For any vector field* $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ *A vector field* $\mathbf{F}$ *is said to be* ***conservative*** *if there exists a function* $f : \mathbb{R}^n \to \mathbb{R}$ *such that*
$$\int_\sigma \mathbf{F} \cdot d\mathbf{s} = \int_\gamma \mathbf{F} \cdot d\mathbf{s}$$

DEFINITION 22.5.

$$\mathbf{F} = \nabla f$$

THEOREM 22.6. *Suppose* $f : \mathbb{R}^n \to \mathbb{R}$ *is differentiable and that* $\sigma : [a,b] \to \mathbb{R}^n$ *be a piecewise differentiable path. Then*

$$\int_\sigma \nabla f \cdot d\mathbf{s} = f(\sigma(b)) - f(\sigma(a))$$

Proof. We have

$$\begin{aligned}
\int_\sigma \nabla f \cdot d\mathbf{s} &\equiv \int_a^b \nabla f \cdot \frac{d\sigma}{dt} dt \\
&= \int_a^b \frac{d}{dt}(f \circ \sigma)\, dt \quad \text{(by the chain rule)} \\
&= f(\sigma(b)) - f(\sigma(a)) \quad \text{(by the Fundamental Theorem of Calculus)}
\end{aligned}$$

DEFINITION 22.7. *A vector field* $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ *is called* ***conservative*** *if* $\mathbf{F} = \nabla f$ *for some function* $f : \mathbb{R}^n \to \mathbb{R}$.

REMARK 22.8. When a force field is conservative, the work done in moving a body from one point to another depends only on the initial and final positions; independent of the path taken. For, in this case,

$$W = \int_\sigma \mathbf{F} \cdot d\mathbf{s} = \int_\sigma \nabla f \cdot d\mathbf{s} = f(\sigma(b)) - f(\sigma(a))$$