LECTURE 21

Integration by a Change of Variables, Cont'd

1. How a Coordinate Changes $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ affects Small Areas

Let us now compare the area $\Delta s \Delta t$ of a small rectangle with the area of its image under a coordinate change

$$\Phi : \mathbb{R}^2 \to \mathbb{R}^2 \, , \, (s,t) \mapsto (u(s,t), v(s,t))$$

Let R be the rectangle in the st-plane with vertices (s,t), $(s+\Delta s, t)$, $(s+\Delta s, t+\Delta t)$, and $(s, t+\Delta t)$. Just like in the example at the end of the previous lecture, we'll determine the shape of $\Phi(R)$ be looking at how the boundary of R is mapped by Φ.

$$\begin{aligned}
\gamma_1(\tau) &= (s+\tau, t) \quad , \quad \tau \in (0, \Delta s) \\
\gamma_2(\tau) &= (s+\Delta s, t+\tau) \quad , \quad \tau \in (0, \Delta t) \\
\gamma_1(\tau) &= (s+\Delta s-\tau, t+\Delta t) \quad , \quad \tau \in (0, \Delta s) \\
\gamma_1(\tau) &= (s, t+\Delta t-\tau) \quad , \quad \tau \in (0, \Delta t)
\end{aligned}$$

The corresponding paths in the uv-plane will be

$$\begin{aligned}
\sigma_1(\tau) &= (u(s+\tau,t), v\,(s+\tau,t)) \quad , \quad \tau \in (0, \Delta s) \\
&= \left(u(s,t) + \frac{\partial u}{\partial s}\tau + \cdots , v(s,t) + \frac{\partial v}{\partial s}\tau + \cdots \right) \quad , \quad \tau \in (0,\Delta s) \\
&\approx \Phi(s,t) + \left(\frac{\partial u}{\partial s}, \frac{\partial v}{\partial s}\tau \right)\tau \quad , \quad \tau \in (0,\Delta s) \\
\sigma_2(\tau) &= (u(s+\Delta t, t+\tau), v\,(s+\Delta s, t+\tau)) \quad , \quad \tau \in (0,\Delta t) \\
&= \left(u(s,t) + \frac{\partial u}{\partial s}\Delta s + \frac{\partial u}{\partial t}\tau \cdots , v(s,t) + \frac{\partial v}{\partial s}\Delta s + \frac{\partial v}{\partial t}\tau \cdots \right) \quad , \quad \tau \in (0,\Delta t) \\
&\approx \Phi(s,t) + \left(\frac{\partial u}{\partial s}, \frac{\partial v}{\partial s} \right)\Delta s + \left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t} \right)\tau \quad , \quad \tau \in (0,\Delta t) \\
\sigma_3(\tau) &= (u(s+\Delta s-\tau, t+\Delta t), v\,(s+\Delta s-\tau, t+\Delta t)) \quad , \quad \tau \in (0,\Delta t) \\
&= \left(u(s,t) + \frac{\partial u}{\partial s}(\Delta s-\tau) + \frac{\partial u}{\partial t}\Delta t \cdots , v(s,t) + \frac{\partial v}{\partial s}\,(\Delta s-\tau) + \frac{\partial v}{\partial t}\Delta t \cdots \right) \quad , \quad \tau \in (0,\Delta t) \\
&= \Phi(s,t) + \left(\frac{\partial u}{\partial s}, \frac{\partial v}{\partial s} \right)(\Delta s-\tau) + \left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t} \right)\Delta t \quad , \quad \tau \in (0,\Delta s) \\
\sigma_4(\tau) &= (u(s, t+\Delta t-\tau), v\,(s, t+\Delta t-\tau)) \quad , \quad \tau \in (0,\Delta t) \\
&= \left(u(s,t) + \frac{\partial u}{\partial t}(\Delta t-\tau) \cdots , v(s,t) + \frac{\partial v}{\partial t}(\Delta t-\tau) + \cdots \right) \quad , \quad \tau \in (0,\Delta t) \\
&= \Phi(s,t) + \left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t} \right)(\Delta t-\tau) \quad , \quad \tau \in (0,\Delta t)
\end{aligned}$$

The corresponding curves are thus the edges of a parallelogram with vertices

$$\begin{aligned}
\mathbf{v}_1 &= \Phi(s,t) \\
\mathbf{v}_2 &= \Phi(s+\Delta s,t) \approx \Phi(s,t) + \left(\frac{\partial u}{\partial s},\frac{\partial v}{\partial s}\right)\Delta s \\
\mathbf{v}_3 &= \Phi(s+\Delta s,t+\Delta t) \approx \Phi(s,t) + \left(\frac{\partial u}{\partial s},\frac{\partial v}{\partial s}\right)\Delta s + \left(\frac{\partial u}{\partial t},\frac{\partial v}{\partial t}\right)\Delta t \\
\mathbf{v}_4 &= \Phi(s,t+\Delta t) \approx \Phi(s,t) + \left(\frac{\partial u}{\partial t},\frac{\partial v}{\partial t}\right)\Delta t
\end{aligned}$$

Recall (see the lecture on the properties of divergence of a vector field) that the area of a parallelogram is equal to the magnitude of the cross product of the vectors representing two adjacent sides. Thus the area of the image of R under the map Φ should be

$$\begin{aligned}
\|(\mathbf{v}_2-\mathbf{v}_1)\times(\mathbf{v}_4-\mathbf{v}_1)\| &= \left\|\left(\frac{\partial u}{\partial s}\Delta s,\frac{\partial v}{\partial s}\Delta s,0\right)\times\left(\frac{\partial u}{\partial t}\Delta t,\frac{\partial v}{\partial t}\Delta t\right)\right\| \\
&= \left\|\left(0,0,\frac{\partial u}{\partial s}\frac{\partial v}{\partial t}\Delta s\Delta t - \frac{\partial u}{\partial t}\frac{\partial v}{\partial s}\Delta s\Delta t\right)\right\| \\
&= \left|\frac{\partial u}{\partial s}\frac{\partial v}{\partial t} - \frac{\partial u}{\partial t}\frac{\partial v}{\partial s}\right|\Delta s\Delta t
\end{aligned}$$

Thus, under the coordinate transformation, the area of a small rectangle R changes by a factor of

$$\left|\frac{\partial u}{\partial s}\frac{\partial v}{\partial t} - \frac{\partial u}{\partial t}\frac{\partial v}{\partial s}\right|$$

This factor is called the **Jacobian** of the coordinate transformation Φ. We often write it as follows

$$\mathbf{J}(\Phi) = \det\left|\begin{array}{cc} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{ds} & \frac{\partial v}{\partial t}\end{array}\right| = \det(\mathbf{D}\Phi)$$

2. Change of Variables Formula

Let me summarize what we have see so far.

1. A coordinate transformation can be regarded as a continuous 1-to-1 map
$$\Phi : \mathbb{R}^2 \to \mathbb{R}^2 : (s,t) \mapsto (u(s,t),v(x,t))$$
2. Such a transformation in general distorts the shapes of regions; that is to say, if R is a region in the st-plane, the shape of its image $\Phi(R)$ may be quite different from that of R. Nevertheless, the shape of $\Phi(R)$ can be easily determined by examining how the boundary curves of R gets mapped by Φ.
3. Not only does the shape of a region R change under the map Φ, its area changes as well. The calculation in the preceding section revealed that if R is an infinitesimal rectangle of area $\Delta s\Delta t$, then $\Phi(R)$ is a parallellogram of area
$$\left|\frac{\partial u}{\partial s}\frac{\partial v}{\partial t} - \frac{\partial u}{\partial t}\frac{\partial v}{\partial s}\right|\Delta s\Delta t$$

Now let R be a larger rectangle in the st-plane and consider how we might compute an integral of a function $f(u,v)$ over $\Phi(R)$. Now

$$\int_{\Phi(R)} f(u,v)\,dudv$$

is in principle the limit of a Riemann sum of the form

$$\sum_{i,j} f(\mathbf{p}_{ij})Area(A_{ij}) \tag{21.1}$$

where the $\{A_{ij}\}$ is a collection of small disjoint subregions such that

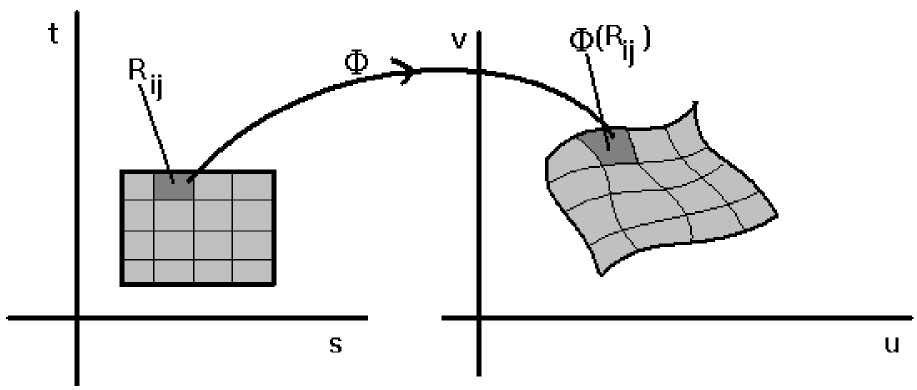
$$\Phi(R) = \bigcup A_{ij}$$

and the points $\mathbf{p}_{ij}$ are arbitrarily chosen points within the corresponding subregions. One way to create such a partitioning of $\Phi(R)$ would be to first partition the original rectangle R into a bunch of smaller rectangles $\{R_{ij}\}$, say of width Δs and height Δt, and then set

$$A_{ij} = \Phi(R_{ij})$$

If we also choose points $(s_i, t_j) \in R_{ij}$ then $\mathbf{p}_{ij} = \Phi(s_i, t_j)$ will lie in A_{ij} and so we can write the Riemann sum (21.1) as

$$\int_{\Phi(R)} f(u,v)dudv = \sum_{i,j} f(\mathbf{p}_{ij}) Area(A_{ij}) = \sum_{i,j} f(\Phi(s_i,t_j)) Area(\Phi(R_{ij}))$$



But as we have seen above if the rectangles R_{ij} are small enough, their images by Φ will be parallelograms of area

$$\left|\frac{\partial u}{\partial s}\frac{\partial v}{\partial t} - \frac{\partial u}{\partial t}\frac{\partial v}{\partial s}\right| \Delta s \Delta t$$

Thus,

$$\int_{\Phi(R)} f(u,v)dudv = \sum_{i,j} f(\Phi(s_i,t_j)) \left|\frac{\partial u}{\partial s}\frac{\partial v}{\partial t} - \frac{\partial u}{\partial t}\frac{\partial v}{\partial s}\right| \Delta s \Delta t$$

But now the right hand side is identifiable as a Riemann sum over the rectangle R. In the limit as $\Delta s, \Delta t \to 0$, we therefore have

$$\int_{\Phi(R)} f(u,v)dudv = \lim_{\Delta s, \Delta t \to 0} \sum_{i,j} f(\Phi(s_i,t_j)) \left|\frac{\partial u}{\partial s}\frac{\partial v}{\partial t} - \frac{\partial u}{\partial t}\frac{\partial v}{\partial s}\right| \Delta s \Delta t \equiv \int_R f(\Phi(s,t)) \left|\frac{\partial u}{\partial s}\frac{\partial v}{\partial t} - \frac{\partial u}{\partial t}\frac{\partial v}{\partial s}\right| dsdt$$

Thus, we arrive at the following formula.

$$\int_{\Phi(R)} f(u,v)dudv = \int_R f(\Phi(s,t)) \left|\frac{\partial u}{\partial s}\frac{\partial v}{\partial t} - \frac{\partial u}{\partial t}\frac{\partial v}{\partial s}\right| dsdt$$

EXAMPLE 21.1. Use the following coordinate transformation

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

to calculate the integral

$$\int_D x^2 dA$$

where D is the disk

$$D = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq R^2\right\}$$

- In terms of the coordinates r and θ the disk D is prescribed by

$$D = \left\{(r, \theta) \in \mathbb{R}^2 \mid 0 \leq r \leq R \, , \, 0 \leq \theta \leq 2\pi\right\}$$

which is a nice rectangular region. The Jacobian of the transfomation

$$\Phi : (r, \theta) \longrightarrow (x = r\cos(\theta), y = r\sin(\theta))$$

is

$$\begin{aligned} J(\Phi) &= \left|\frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial y}{\partial r}\frac{\partial x}{\partial \theta}\right| \\ &= \left|(\cos(\theta))\,(r\cos(\theta)) - (-r\sin(\theta))\,(\sin(\theta))\right| \\ &= \left|r\left(\cos^2(\theta) + \sin^2(\theta)\right)\right| \\ &= |r| \\ &= r \end{aligned}$$

So

$$\begin{aligned} \int_D x^2 dA &= \int_0^R \int_0^{2\pi} (x(r, \theta))^2 J(\Phi)\, d\theta dr \\ &= \int_0^R \int_0^{2\pi} \left(r^2 \cos^2\theta\right) r d\theta dr \\ &= \int_0^R r^3 \left(\frac{\theta}{2} + \frac{1}{2}\cos(\theta)\sin(\theta)\right)\Bigg|_0^{2\pi} dr \\ &= \pi \int_0^R r^3 dr \\ &= \frac{\pi}{4} R^4 \end{aligned}$$