LECTURE 20

# Integration by a Change of Variables

## 1. Introduction

Consider the region $B \subset \mathbb{R}^3$ defined by

$$B = \left\{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq r^2 \right\}$$

This is just a ball of radius $r$. According the theory developed last time, we can integrate a function $f(x, y, z)$ over $B$ by first realizing $B$ as an *elementary region* in $\mathbb{R}^3$

$$B = \left\{ (x, y, z) \in \mathbb{R}^3 \mid -r \leq x \leq r \, , \, -\sqrt{r^2 - x^2} \leq y \leq \sqrt{r^2 - x^2} \, , \, -\sqrt{r^2 - x^2 - y^2} \leq z \leq \sqrt{r^2 - x^2 - y^2} \right\}$$

and then calculating

$$\int_B f(x, y, z) dV = \int_{-r}^{r} \int_{-\sqrt{r^2 - x^2}}^{\sqrt{r^2 - s^2}} \int_{-\sqrt{r^2 - x^2 - y^2}}^{\sqrt{r^2 - s^2 - y^2}} f(x, y, z) \, dz \, dy \, dx$$

A major complication in carrying out the multiple integration on the right-hand-side is the fact that the limits of integration are, in the first two steps, functions rather than simple constants. It certainly would make life easier if we could instead integrate $f(x, y, z)$ over a rectangular box.

But note that the ball $B$ can also be described in terms of spherical coordinates as

$$B = \left\{ (\rho, \theta, \phi) \in \mathbb{R}^3 \mid 0 \leq \rho \leq r \, , \, 0 \leq \theta \leq 2\pi \, , \, 0 \leq \phi \leq \pi \right\}$$

Thus, if we choose to parameterize the ball $B$ in terms of spherical coordinates, integration over $B$ would correspond to integration over a rectangular box in $(\rho, \theta, \phi)$ space. Unfortunately,

$$\int_B f(x, y, z) dV \neq \int_0^r \int_0^{2\pi} \int_0^{\pi} f(\rho, \theta, \phi) \, d\phi \, d\theta \, d\rho \tag{20.1}$$

but the idea that integrations can be easier if we change variables is actually a practical one.

To see why (20.1) can not be the right formula, consider the change of variables formula for functions of a single variable. If $u(x)$ is 1:1 function and $x(u)$ is its inverse then

$$x = x(u) \quad \Rightarrow \quad dx = \frac{dx}{du} du$$

and

$$\int_a^b f\left(u(x)\right) dx = \int_{u(a)}^{u(b)} f(u) \frac{dx}{du} du$$

Thus, when we change variables of integration, we don't simply substitute a new variable $u$ for the expression $u(x)$, we also insert an additional factor$\frac{dx}{du}$ into the integrand. What this factor does is keep track of the relative length scales of the variables $x$ and $u$: if $x$ changes by a small amount $\Delta x$ then this length in terms of $u$ is

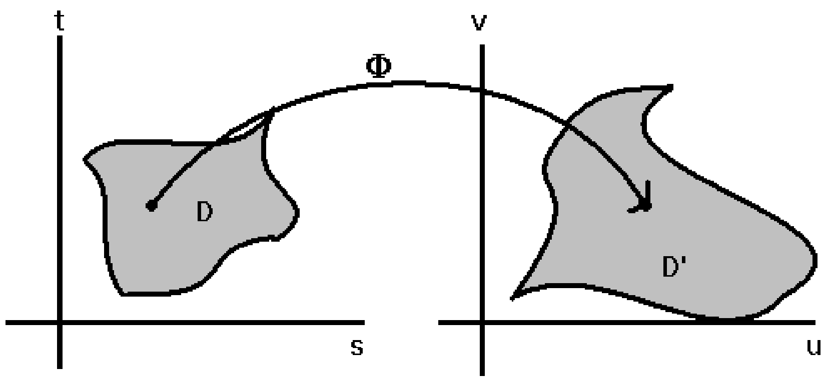
$$\Delta x = \frac{\Delta u}{\Delta u} \Delta x = \frac{\Delta x}{\Delta u} \Delta u \sim \frac{dx}{du} \Delta u$$

We shall have the same sort of phenomenon when we change variables for integrations over two dimensional regions: we shall have to insert a scaling factor that relates the infinitesimal area $\Delta x \Delta y$ with respect to one set of coordinates to the infinitesimal area $\Delta u \Delta v$ corresponding to the new set of coordinates.

## 2. The Geometry of Maps from $\mathbb{R}^2$ to $\mathbb{R}^2$

We shall first look at what happens when a region in the plane is mapped to another region by means of a change of coordinates. Let us denote by $s$ and $t$ the variables representing the original coordinates and by $u$ and $v$ the new coordinates (perhaps, $s = x$, $t = y$, the usual rectangular coordinates and $u = r$, $v = \theta$, the usual polar coordinates). We shall think of such a transformation as a function $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ and we shall assume that function is continuous, 1 to 1, and onto meaning

- The image of any smooth connected region $D$ in the $st$-plane is a smooth connected region $D'$ in the $uv$-plane.
- Given any point $(u, v) \in D'$ there is only one point $(s, t)$ in $D$ such that $(u, v) = \Phi(s, t)$.
- If $(s, t)$ is a point on the boundary of $D$, then $\Phi(s, t)$ is a point on the boundary of $D'$.




Example 20.1. Consider the coordinate transformation $\Phi : (s, t) \to (u, v)$ defined by

$$\begin{aligned} u &= u(s,t) = s^2 - t^2 \\ v &= v(s,t) = st \end{aligned}$$

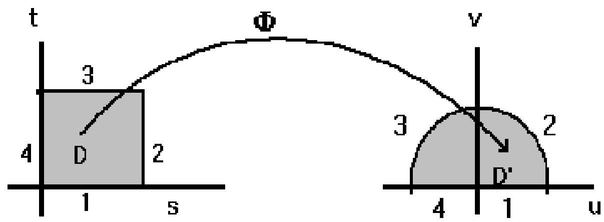
Find the image of the square

$$D = \left\{(s,t) \in \mathbb{R}^2 \mid 0 \le s \le 1 \,,\, 0 \le t \le 1\right\}$$

- The important thing to remember about coordinate transformations is that boundaries always get mapped onto boundaries; so if you can figure out how the boundaries get mapped you can see what the entire image is.

  Now the boundary of the square $D$ consists of four line segments

$$\begin{aligned} \gamma_1(t) &= (t, 0) \quad , \quad t \in (0,1) \\ \gamma_2(t) &= (1, t) \quad , \quad t \in (0,1) \\ \gamma_3(t) &= (1-t, 1) \quad , \quad t \in (0,1) \\ \gamma_4(t) &= (0, 1-t) \quad , \quad t \in (0,1) \end{aligned}$$




The boundary of $D'$ will be the image of these four line segments under the map $\Phi : (s,t) \mapsto \left(s^2 - t^2, st\right)$

$$\begin{aligned}
\sigma_1(t) &= \Phi(\gamma_1(t)) = \left(t^2 - 0^2, 0\right) = (t^2, 0) \quad , \quad t \in (0,1) \\
\sigma_2(t) &= \Phi(\gamma_2(t)) = \left(1 - t^2, t\right) \quad , \quad t \in (0,1) \\
\sigma_3(t) &= \Phi(\gamma_3(t)) = \left((1-t)^2 - 1^2, (1-t)\right) \quad , \quad t \in (0,1) \\
&= (\tau^2 - 1, \tau) \quad , \quad \tau \in (1,0) \qquad (\text{here } \tau = 1 - t) \\
\sigma_4(t) &= \Phi(\gamma_4(t)) = \left(0 - (1-t)^2, 0\right) \quad , \quad t \in (0,1) \\
&= (-\tau^2, 0) \quad , \quad \tau \in (0,1)
\end{aligned}$$

If we now plot these four curves we see

1. The image of $\sigma_1$ is the line segment from $(0,0)$ to $(1,0)$.
2. The image of $\sigma_2$ is the piece of the parabola $u = 1 - v^2$ between $(1,0)$ and $(0,1)$.
3. The image of $\sigma_3$ is the piece of the parabola $u = v^2 - 1$ between $(0,1)$ and $(-1,0)$.
4. The image of $\sigma_4$ is the line segment from $(-1,0)$ to $(0,0)$.

Thus, $D'$ looks like

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