LECTURE 19

Integrals over 3-Dimensional Regions

1. Integrals over Rectangular Boxes

The definition of an integral over a 3-dimensional rectangular box is a straight-forward generalization of the definition of an integral over a (2-dimensional) rectangle:

Definition 19.1. *A Riemann sum of a function $f : \mathbb{R}^3 \to \mathbb{R}$ over a rectangular box*

$$R = \left\{ (x, y, z) \in \mathbb{R}^3 \mid a \leq x \leq b\,,\ c \leq y \leq d\,,\ e \leq z \leq f \right\}$$

is a sum of the form

$$S_n = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} f\left(\mathbf{p}_{ijk}\right) \phi \Delta x \Delta y \Delta z$$

where

$$\begin{aligned} \Delta x &= \frac{b-a}{n} \\ \Delta y &= \frac{d-c}{n} \\ \Delta z &= \frac{f-e}{n} \end{aligned}$$

and $\mathbf{p}_{ijk}$ is a point within the rectangular box

$$R_{ijk} = \left\{ (x, y, z) \in \mathbb{R}^3 \mid x_{i-1} \leq x \leq x_i\,,\ y_{j-1} \leq y \leq y_j\,,\ z_{k-1} \leq z \leq z_k \right\}$$

where

$$\begin{aligned} x_i &= a + i\Delta x \\ yj &= c + j\Delta y \\ z_k &= e + k\Delta z \end{aligned}$$

Definition 19.2. *The integral of a function $f : \mathbb{R}^3 \to \mathbb{R}$ over a rectangular box R is the limit of a sequence of Riemann sums of f over R*

$$\int_R f(x, y, z) dV = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} f\left(\mathbf{p}_{ijk}\right) \phi \Delta x \Delta y \Delta z$$

whenever this limit exists and is independent of the choice of points $\mathbf{p}_{ikj}$.

Theorem 19.3. *If $f : \mathbb{R}^3 \to \mathbb{R}$ is continuous on R then*

$$\int_R f(x, y, z) dV$$

exists and

$$\int_R f(x, y, z) dV = \int_a^b \left(\int_c^d \left(\int_e^f f(x, y, z) dz \right) dy \right) dx$$

Moreover, its value is independent of the order of integration on the right hand side.

EXAMPLE 19.4. Evaluate

$$\int_R xyz\, dV$$

where R is the rectangular box

$$R = \left\{(x, y, z) \in \mathbb{R}^3 \mid -1 \leq x \leq 1\,,\, 0 \leq y \leq 2\,,\, 0 \leq z \leq 1\right\}$$

- We have

$$\begin{aligned}
\int_R xyz\, dV &= \int_{-1}^{1} \left(\int_0^2 \left(\int_0^1 xyz\, dz \right) dy \right) dx \\
&= \int_{-1}^{1} \left(\int_0^2 \left(\frac{1}{2} yx - 0 \right) dy \right) dx \\
&= \int_{-1}^{1} \left(\frac{1}{4} x(2^2) - 0 \right) dx \\
&= \frac{1}{2}(1)^2 - \frac{1}{2}(-1)^2 \\
&= 0
\end{aligned}$$

2. Integrals over More General Regions

DEFINITION 19.5. *By an* ***elementary region*** *in* $\mathbb{R}^3$ *we shall mean a region that can be prescribed by*

- *restricting one coordinate, say* x_3*, to lie between the graphs of two functions of the other coordinates*

$$\psi_1(x_1, x_2) \leq x_3 \leq \psi_2(x_1, x_2)$$

- *restricting a second coordinate, say* x_2*, to lie betweeen the graphs of two functions of the remaining coordinate*

$$\phi_1(x_1) \leq x_2 \leq \phi_2(x_1)$$

- *restricting the last coordinate to be lie between two constants*

$$a \leq x_1 \leq b$$

THEOREM 19.6. *Suppose* S *is an elementary region in* $\mathbb{R}^3$ *and* $f(x, y, z)$ *is continuous on* S*. Then*

$$\int_S f(x, y, z) dV = \int_a^b \left(\int_{\phi_1(x_1)}^{\phi_2(x_1)} \left(\int_{\psi_1(x_1, x_2)}^{\psi_2(x_1, x_2)} f(x_1, x_2, x_3)\, dx_3 \right) dx_2 \right) dx_1$$

EXAMPLE 19.7. Let B be a ball of radius 1 centered at the origin. Compute

$$\int_B dV$$

- We can realize the ball as an elementary region as follows.

$$B = \left\{ -1 \leq x \leq 1\,, -\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}\,,\, -\sqrt{1 - x^2 - y^2} \leq z \leq \sqrt{1 - x^2 - y^2} \right\}$$

So

$$\begin{aligned}
\int_B dV &= \int_{-1}^{1} \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz \right) dy \right) dx \\
&= \int_{-1}^{1} \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(2\sqrt{1 - x^2 - y^2} \right) dy \right) dx
\end{aligned}$$

Using the identity

$$\int_{-a}^{a} \sqrt{a^2 - y^2}\,dy = \frac{a^2}{2}\pi$$

we have

$$\begin{aligned}
\int_B dV &= \int_{-1}^{1} 2\left(\frac{\left(1-x^2\right)}{2}\pi\right) \\
&= \pi\left(x - \frac{x^3}{3}\right)\bigg|_{-1}^{1} \\
&= \pi\left(1 - \frac{1}{3} - \left(-1 + \frac{1}{3}\right)\right) \\
&= \frac{4}{3}\pi
\end{aligned}$$