Reversing the Order of Integration

Last time I presented the following theorem.

**Theorem 18.1.** If $S$ is a region of Type I, then

$$
\int_S f(x,y)\,dA = \int_a^b \left( \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy \right) \,dx
$$

If $S$ is a region of Type II, then

$$
\int_S f(x,y)\,dA = \int_c^d \left( \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx \right) \,dy
$$

I should point out that, unlike the case of integrals over rectangles, there is only one order in which we can carry out the integrations. If $S$ is a Type I region we have to integrate over $y$ before we integrate over $x$ and if $S$ is a Type II region we have to carry out the integration over $x$ before we integrate over $y$.

However, as remarked last time, sometimes a region is both Type I and Type II. This does not mean that

$$
\int_a^b \left( \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy \right) \,dx = \int_{\phi_1(x)}^{\phi_2(x)} \left( \int_a^b f(x,y)\,dx \right) \,dy
$$

because the right hand side doesn’t really make any sense. Rather, it means that we can prescribe the region $S$ in two different ways

$$
S = \{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b, \ \phi_1(x) \leq y \leq \phi_2(x)\}
$$

$$
= \{(x,y) \in \mathbb{R}^2 \mid \psi_1(y) \leq x \leq \psi_2(y), c \leq y \leq d\}
$$

The preceding theorem applied to this situation simply says

$$
\int_a^b \left( \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy \right) \,dx = \int_c^d \left( \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx \right) \,dy
$$

In order to reverse the order of integration of an integral like

$$
\int_a^b \left( \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy \right) \,dx
$$

one therefore first has to figure out how to parameterize the region of integration

$$
\{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b, \ \phi_1(x) \leq y \leq \phi_2(x)\}
$$

as a Type II region; that is to say, one has to figure out what $c, d, \psi_1(y)$, and $\psi_2(y)$ are.

**Example 18.2.** Reverse the order of integration of

$$
\int_0^1 \int_x^1 xy \,dy \,dx \equiv \int_0^1 \left( \int_0^x xy \,dy \right) \,dx
$$
The first thing we need to do is figure out what the region of integration looks like. Evidently

\[ S = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, \; x \leq y \leq 1\} \]

This same region can also be described as

\[ S = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x < y, \; 0 \leq y \leq 1\} \]

Therefore,

\[ \int_0^1 \int_x^1 xy \, dy \, dx = \int_0^1 \int_0^x xy \, dx \, dy \]

\[ \square \]

**Example 18.3.** Reverse the order of integration of

\[ \int_0^{\pi/2} \int_0^{\cos(\theta)} \cos(\theta) \, dr \, d\theta \]

The region of integration in this example looks like

This region can also be described as

\[ \{(r, \theta) \mid 0 \leq r \leq 1, \; 0 \leq \theta \leq \cos^{-1}(r)\} \]

so

\[ \int_0^{\pi/2} \int_0^{\cos(\theta)} \cos(\theta) \, dr \, d\theta = \int_0^1 \int_0^{\cos^{-1}(r)} \cos(\theta) \, d\theta \, dr \]

\[ \square \]

**Example 18.4.** Reverse the order of integration of

\[ \int_0^2 \int_0^{2-y} (x+y)^2 \, dx \, dy \]
• The region of integration in this example looks like

\[
\begin{align*}
\int_0^2 \int_0^{2 - y} (x + y)^2 \, dx \, dy &= \int_0^1 \int_0^{2 - y} (x + y)^2 \, dy \, dx \\
\end{align*}
\]

Example 18.5. Integrate

\[
\int_D e^{-x-y} \, dA
\]

where \( D \) is the interior of the triangle with vertices \((0,0), (1,3), \) and \((2,3)\).

• The region of integration in this example looks like

and is most easily represented as a Type II region. Clearly, \( y \) ranges between 0 and 3. The boundary on the left hand side is a straight line passing through the origin and has slope

\[
\frac{\Delta y}{\Delta x} = \frac{3}{1} = 3
\]

and so it coincides with the line

\[
y = 3x \quad \Rightarrow \quad x = \frac{1}{3} y
\]

The boundary on the left is a straight line passing through the origin with slope

\[
\frac{\Delta y}{\Delta x} = \frac{3}{2}
\]

and so it coincides with the line

\[
y = \frac{3}{2} x \quad \Rightarrow \quad x = \frac{2}{3} y
\]
Thus the triangular region $D$ is

$$D = \{(x, y) \mid \frac{1}{3} y \leq x \leq \frac{2}{3} y, 0 \leq y \leq 3\}$$

Hence,

$$\int_D e^{-x-y} dA = \int_0^3 \int_{\frac{x}{y}}^{\frac{2y}{y}} e^{-x-y} \, dx \, dy$$

$$= \int_0^3 -e^{-x-y}\bigg|_{\frac{x}{y}}^{\frac{2y}{y}} \, dx$$

$$= \int_0^3 \left(-e^{-\frac{3}{2}y} + e^{-\frac{5}{3}y}\right) \, dy$$

$$= \frac{3}{5} e^{-\frac{3}{2}y} - \frac{3}{4} e^{-\frac{5}{3}y}\bigg|_0^3$$

$$= \frac{3}{5} e^{-5} - \frac{3}{4} e^{-4} - \frac{3}{5} + \frac{3}{4}$$

\[\square\]

**Example 18.6.** Integrate

$$\int_D (1 + xy) \, dA$$

where $D$ is the region

$$D = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 2, y \geq 0\}$$

- The region of integration looks like \[\square\]