LECTURE 18

Reversing the Order of Integration

Last time I presented the following theorem.

THEOREM 18.1. *If S is a region of Type I, then*

$$\int_S f(x,y)dA = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy \right) dx$$

If S is a region of Type II, then

$$\int_S f(x,y)dA = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx \right) dy$$

I should point out that, unlike the case of integrals over rectangles, there is only one order in which we can carry out the integrations. If S is a Type I region we have to integrate over y before we integrate over x and if S is a Type II region we have to carry out the integration over x before we integrate over y.

However, as remarked last time, sometimes a region is both Type I and Type II. This does not mean that

$$\int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy \right) dx = \int_{\phi_1(x)}^{\phi_e(x)} \left(\int_a^b f(x,y)dx \right) dy$$

because the right hand side doesn't really make any sense. Rather, it means that we can prescribe the region S in two different ways

$$\begin{aligned} S &= \left\{ (x,y) \in \mathbb{R}^2 \mid a \leq x \leq b \ , \ \phi_1(x) \leq y \leq \phi_2(x) \right\} \\ &= \left\{ (x,y) \in \mathbb{R}^2 \mid \psi_1(y) \leq x \leq \psi_2(y) \ , \ c \leq y \leq d \right\} \end{aligned}$$

The preceding theorem applied to this situation simply says

$$\int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy \right) dx = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx \right) dy$$

In order to reverse the order of integration of an integral like

$$\int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy \right) dx$$

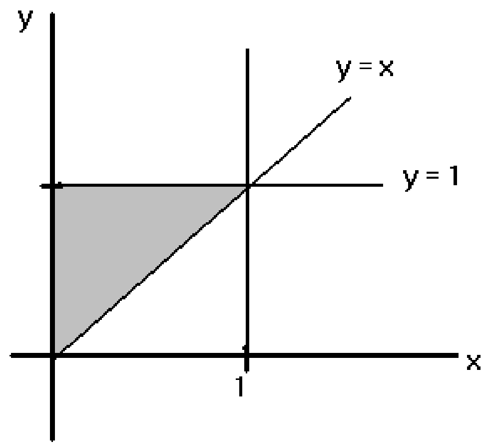
one therefore first has to figure out how to parameterize the region of integration

$$\left\{ (x,y) \in \mathbb{R}^2 \mid a \leq x \leq b \ , \ \phi_1(x) \leq y \leq \phi_2(x) \right\}$$

as a Type II region; that is to say, one has to figure out what $c, d, \psi_1(y)$, and $\psi_2(y)$ are.

EXAMPLE 18.2. Reverse the order of integration of

$$\int_0^1 \int_x^1 xy \, dy \, dx \equiv \int_0^1 \left(\int_x^0 xy \, dy \right) dx$$

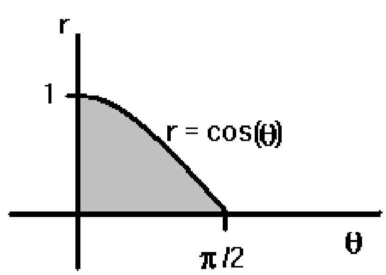


- The first thing we need to do is figure out what the region of integration looks like. Evidently
$$S = \left\{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1\,,\, x \leq y \leq 1\right\}$$
This same region can also be described as
$$S = \left\{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq y\,,\, 0 \leq y \leq 1\right\}$$
Therefore,
$$\int_0^1 \int_x^1 xy\,dy\,dx = \int_0^1 \int_0^x xy\,dx\,dy$$

□

EXAMPLE 18.3. Reverse the order of integration of
$$\int_0^{\pi/2} \int_0^{\cos(\theta)} \cos(\theta)\,dr\,d\theta$$

- The region of integration in this example looks like

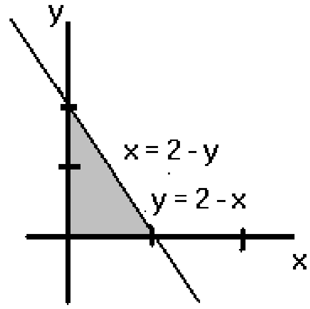


This region can also be described as
$$\left\{(r,\theta) \mid 0 \leq r \leq 1\,,\, 0 \leq \theta \leq \cos^{-1}(r)\right\}$$
so
$$\int_0^{\pi/2} \int_0^{\cos(\theta)} \cos(\theta)\,dr\,d\theta = \int_0^1 \int_0^{\cos^{-1}(r)} \cos(\theta)\,d\theta\,dr$$

□

EXAMPLE 18.4. Reverse the order of integration of
$$\int_0^2 \int_0^{2-y} (x+y)^2 dx\,dy$$

- The region of integration in this example looks like



and it can also be described as a Type I region as

$$\{(x,y) \mid 0 \leq x \leq 1\,,\ 0 \leq y \leq 2-x\}$$

and so

$$\int_0^2 \int_0^{2-y} (x+y)^2 dx\, dy = \int_0^1 \int_0^{2-x} (x+y)^2 dy\, dx$$

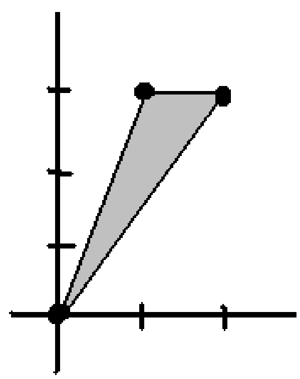
$\square$

Example 18.5. Integrate

$$\int_D e^{-x-y} dA$$

where D is the interior of the triangle with vertices $(0,0)$, $(1,3)$, and $(2,3)$.

- The region of integration in this example looks like



and is most easily represented as a Type II region. Clearly, y ranges between 0 and 3. The boundary on the left hand side is a straight line passing through the origin and has slope

$$\frac{\Delta y}{\Delta x} = \frac{3}{1} = 3$$

and so it coincides with the line

$$y = 3x \quad \Rightarrow \quad x = \frac{1}{3} y$$

The boundary on the left is a straight line passing through the origin with slope

$$\frac{\Delta y}{\Delta x} = \frac{3}{2}$$

and so it coincides with the line

$$y = \frac{3}{2} x \quad \Rightarrow \quad x = \frac{2}{3} y$$

Thus the triangular region D is

$$D = \left\{ (x,y) \mid \frac{1}{3}y \le x \le \frac{2}{3}y\,,\, 0 \le y \le 3 \right\}$$

Hence,

$$\begin{aligned}
\int_D e^{-x-y} dA &= \int_0^3 \int_{\frac{1}{3}y}^{\frac{2}{3}y} e^{-x-y}\, dx\, dy \\
&= \int_0^3 -e^{-x-y}\Big|_{\frac{y}{3}}^{\frac{2y}{3}}\, dx \\
&= \int_0^3 \left(-e^{-\frac{5}{3}y} + e^{-\frac{4}{3}y} \right) dy \\
&= \frac{3}{5}e^{-\frac{5}{3}y} - \frac{3}{4}e^{-\frac{4}{3}y}\Big|_0^3 \\
&= \frac{3}{5}e^{-5} - \frac{3}{4}e^{-4} - \frac{3}{5} + \frac{3}{4}
\end{aligned}$$

□

Example 18.6. Integrate

$$\int_D (1+xy) dA$$

where D is the region

$$D = \left\{ (x,y) \in \mathbb{R}^2 \mid 1 \le x^2 + y^2 \le 2\,,\, y \ge 0 \right\}$$

- The region of integration looks like □