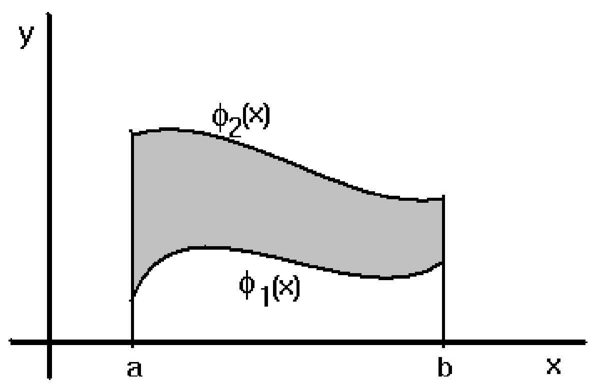
LECTURE 17

Double Integrals over More General Regions

We shall now develop the theory of integration over regions other than rectangles. To do this we will first define regions of Type I, Type II, and Type III and describe how we integrate over these sorts of regions. Integration over more general regions can then be carried out by decomposing the region in question into a disjoint union of Type I, II or III regions, integrating over each subregion, and then summing up the results to get a result for integral over the original region.

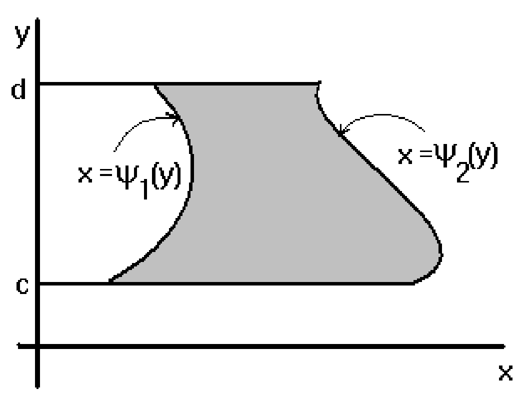
DEFINITION 17.1. *A region $S \subset \mathbb{R}^2$ is called* ***Type I*** *if its boundary consists of two constant values of x and the graphs of two functions of x:*

$$S_I = \left\{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b \;,\; \phi_1(x) \leq y \leq \phi_2(x)\right\}$$

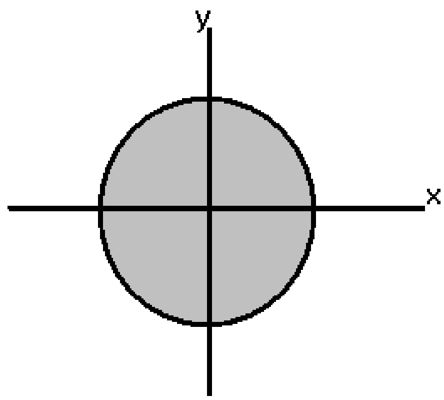


DEFINITION 17.2. *A region $S \subset \mathbb{R}^2$ is called* ***Type II*** *if its boundary consists of two constant values of y and the graphs of two functions of y:*

$$S_{II} = \left\{(x,y) \in \mathbb{R}^2 \mid \psi_1(y) \leq x \leq \psi_2(y) \;,\; c \leq y \leq d\right\}$$



REMARK 17.3. A region S can be both Type I and Type II. For example, a circle has this property.



$$S = \left\{(x,y) \in \mathbb{R}^2 \mid -r \le x \le r \, , \, -\sqrt{r^2 - x^2} \le y \le \sqrt{r^2 - x^2}\right\}$$
$$= \left\{(x,y) \in \mathbb{R}^2 \mid -\sqrt{r^2 - y^2} \le x \le \sqrt{r^2 - y^2} \, , \, -r \le y \le r\right\}$$

DEFINITION 17.4. *If S is a region of Type I or II, then choose a rectangle R in $\mathbb{R}^2$ containing S. Given a continuous function $f : S \to \mathbb{R}$, we define*

$$\int_S f(x,y)dA$$

*the **integral of** f **over** S as follows:*

1. *Extend f to a function $\tilde{f}$ on the rectangle R by setting*
$$\tilde{f}(x,y) = \begin{cases} f(x,y) & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$
2. *The boundary of S then coincides with the discontinuities of $\tilde{f}$ and so because the discontinuities of $\tilde{f}$ are just a finite collection of graphs of continuous functions, $\tilde{f}$ is integrable over R.*
   We can therefore define
$$\int_S f(x,y)dA = \int_R \tilde{f}(x,y)dA$$
   since the right hand side exists.

THEOREM 17.5. *If S is a region of Type I, then*

$$\int_S f(x,y)dA = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy\right) dx$$

If S is a region of Type II, then

$$\int_S f(x,y)dA = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx\right) dy$$

EXAMPLE 17.6. Calculate

$$\int_S dA$$

where S is the disk $\left\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1\right\}$.

- We can present S as a Type I region as follows

$$S = \left\{ (x,y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1 \ , \ -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2} \right\}$$

and so by the preceding theorem

$$\begin{aligned}
\int_S dA &= \int_{-1}^{1} \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \right) dx \\
&= \int_{-1}^{1} \left[\sqrt{1-x^2} - \left(-\sqrt{1-x^2} \right) \right] dx \\
&= \int_{-1}^{1} 2\sqrt{1-x^2} dx \\
&= \left(x\sqrt{1-x^2} + \sin^{-1}(x) \right) \Big|_{-1}^{1} \\
&= 0 + \frac{\pi}{2} - \left(0 - \frac{\pi}{2} \right) \\
&= \pi
\end{aligned}$$