LECTURE 16

Double Integrals over Rectangles

For the last 10 lectures or so we have been developing the calculus of derivatives for functions of several variables. We now turn our attention to the corresponding theory of integrals. We shall begin a definition and discussion of integrals of functions of two variables over a rectangular region within their domain.

Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables and let

$$R = \left\{ (x, y) \in \mathbb{R}^2 \mid a \leq x \leq b \quad , \quad c \leq y \leq d \right\}$$

be a rectangular region entirely contained within the domain U of f. Let n be any integer greater than 1 and set

$$\begin{aligned} \Delta x &= \frac{b-a}{n} \\ \Delta y &= \frac{d-c}{n} \\ x_i &= a + i\Delta x \quad , \quad i = 1, 2, \ldots, n \\ y_j &= c + j\Delta y \quad , \quad j = 1, 2, \ldots, n \end{aligned}$$

Note that

$$\begin{aligned} x_0 &= a \\ x_n &= b \\ y_0 &= c \\ y_n &= d \end{aligned}$$

Let R_{ij}, $i, j = 1, \ldots, n$, be the rectangular subset of R defined by

$$R_{ij} = \left\{ (x, y) \in \mathbb{R}^2 \mid x_{i-1} \leq x \leq x_i \quad , \quad y_{j-1} \leq y \leq y_j \right\}$$

and for each of these rectangles choose a point $\mathbf{p}_{ij} \in R_{ij}$.

Definition 16.1. *An n^{th} order **Riemann sum** of f over the rectangle R is any sum of the form*

$$S_n = \sum_{i=0}^{n} \sum_{j=0}^{n} f(\mathbf{p}_{ij}) \, \Delta x \Delta y$$

Definition 16.2. *Let $\{S_n\}$ be a sequence of Riemann sums of a function f over a rectangle $R \subset \mathbb{R}^2$. If the sequence $\{S_n\}$ converges to a limit as $n \to \infty$ and that limit is independent of the choice of the points $\mathbf{p}_{ij} \in R_{ij}$, then we say that f is **integrable over** R, and we write*

$$\int_R f dA = \lim_{n \to \infty} \left[\sum_{i=0}^{n} \sum_{j=0}^{n} f(\mathbf{p}_{ij}) \, \Delta x \Delta y \right]$$

Theorem 16.3. ***(Properties of Integrals over Rectangles)*** *Let f and $g$be a functions that is integrable over a rectangle R. Then*

1. $\int_R (f+g) dA = \int_R f dA + \int_R g dA$
2. *If c is a constant* $\int_R (cf) dA = c \int_R f dA$

3. *If R is the disjoint union of two rectangular subsets R_1 and R_2 then*

$$\int_R f dA = \int_{R_1} f dA + \int_{R_2} f dA$$

In order to make use of this definition, we need to know when functions are integrable and how to compute these limits of Riemann sums.

THEOREM 16.4. *If f is a continuous function defined on a closed rectangle R then f is integrable.*

We can even integrate discontinuous functions if their discontinuities are not too exotic.

THEOREM 16.5. *Let $f : R \rightarrow \mathbb{R}$ be a bounded real-valued function over a rectangle R, and suppose that the set of points where f is discontinuous is on a finite set of graphs (i.e. subsets of the form $y = \phi_i(x)$ or $x = \psi_j(x)$), of continuous functions. Then f is integrable.*

The following theorem tells how to compute double integrals over rectangular regions.

THEOREM 16.6. ***(Fubini's Theorem).*** *Let f be a continuous function on a rectangular domain*

$$R = \left\{ (x, y) \in \mathbb{R}^2 \mid a \leq x \leq b \quad , \quad c \leq y \leq d \right\}$$

Then

$$\int_R f dA = \int_a^b \left(\int_c^d f(x, y) dy \right) dx = \int_c^d \left(\int_a^b f(x, y) dx \right) dy$$

The integrals on the right hand side of the formula above are just ordinary integrals with respect to a single variable. In other words,

$$\int_c^d \left(\int_a^b f(x, y) dx \right) dy$$

is computed by regarding y as a fixed parameter and integrating

$$\int_a^b f(x, y) dx$$

with respect to x and then integrating the result of this calculation (which of course depends on the parameter y) with respect to y.

EXAMPLE 16.7. Calculate the following integral

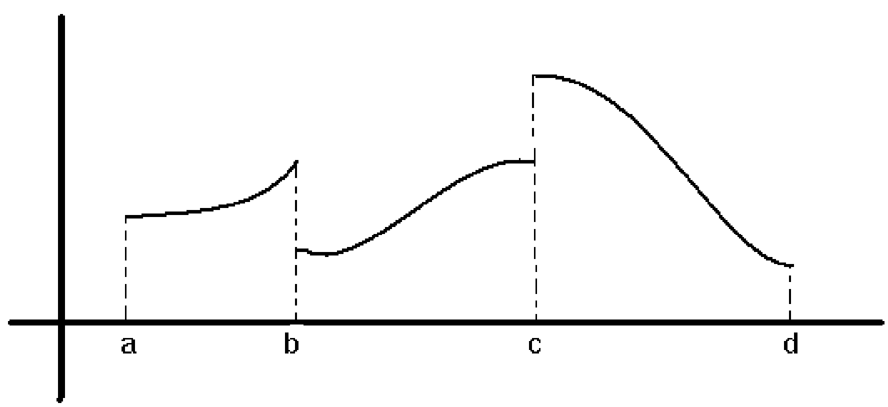
$$\int_R \left(x^2 + y^2 \right) dA$$

where R is the rectangle $0 \leq x \leq 1$, $1 \leq y \leq 2$.

- We have

$$\begin{aligned}
\int_R \left(x^2+y^2\right) dA &= \int_0^1 \left(\int_1^2 \left(x^2+y^2\right) dy \right) dx \\
&= \int_0^1 \left(x^2 y + \frac{1}{3} y^3 \right) \Big|_1^2 dx \\
&= \int_0^1 \left(2x^2 - x^2 + \frac{8}{3} - \frac{1}{3} \right) dx \\
&= \int_0^1 \left(x^2 + \frac{7}{3} \right) dx \\
&= \left(\frac{1}{3} x^3 + \frac{7}{3} x \right) \Big|_0^1 \\
&= \frac{1}{3} + \frac{7}{3} - 0 - 0 \\
&= \frac{8}{3}
\end{aligned}$$

Now recall that for functions of a single variable continuity is not really a necessary condition for the integrability of a function $f(x)$. All we really need is for $f(x)$ to be bounded and possess only a finite number of discontinuities. Because for such functions we compute integrals by summing up the integrals over the various subintervals where $f(x)$ is continuous: for example, for a function $f(x)$ with discontinuities at $x = b$ and $x = c$



we have

$$\int_a^d f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx + \int_c^d f(x)dx$$

We have a analogous situation for discontinuous functions of two variables:

THEOREM 16.8. ***(Generalized Fubini's Theorem).*** *Let $f(x, y)$ be a bounded function with domain R, a rectangle in $\mathbb{R}^2$, and suppose that the discontinuities of f form a finite unition of graphs of continuous functions. If*

$$\int_c^d f(x, y)dy$$

exists for each $x \in R$ (note, $f(x,y)$ need not be continuous along y) then

$$\int_a^b \left(\int_c^d f(x,y)dy \right) dx$$

exists and

$$\int_a^b \left(\int_c^d f(x,y)dy \right) dx = \int \int_R f dA$$

Example 16.9. Calculate

$$\int_R |xy| \ dA$$

where R is the rectangle $0 \leq x \leq 2$, $-1 \leq y \leq 1$.

- The function $f(x,y) = |xy|$ is not really discontinuous; however, its formula in terms of the variables x and y depends on the sign of xy. Since x is always positive within the rectangle R, we have

$$f(x,y) = |xy| = \begin{cases} -xy & -1 \leq y < 0 \\ xy & 0 \leq y \leq 1 \end{cases}$$

  Thus, by Fubini's Theorem

$$\begin{aligned} \int_R f(x,y)dA &= \int_0^2 \left(\int_{-1}^1 f(x,y)dy \right) dx \\ &= \int_0^2 \left(\int_{--1}^0 f(x,y)dy + \int_0^1 f(x,y)dy \right) dx \\ &= \int_0^2 \left(\int_{--1}^0 -xy \ dy + \int_0^1 xy \ dy \right) dx \\ &= \int_0^2 \left(-\frac{1}{2}xy^2 \bigg|_{-1}^0 + \frac{1}{2}xy^2 \bigg|_0^1 \right) dx \\ &= \int_0^2 \left(0 + \frac{1}{2}x + \frac{1}{2}x - 0 \right) dx \\ &= \int_0^2 x dx \\ &= \frac{1}{2}x^2 \bigg|_0^2 \\ &= \frac{1}{2}(4-0) \\ &= 2 \end{aligned}$$