LECTURE 14

The Divergence and Curl of a Vector Field

1. The Divergence of a Vector Field

DEFINITION 14.1. *Let* $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ *be a differentiable vector field. The **divergence** of* $\mathbf{F}$*, denoted* $\nabla \cdot \mathbf{F}$*, is the real-valued function on* $\mathbb{R}^n$ *defined by*

$$\nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_n}{\partial x_n}$$

EXAMPLE 14.2. If $\mathbf{F}(x,y,z) = \left(x^2 + y^2, z^2, xyz\right)$ then

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}\left(x^2 + y^2\right) + \frac{\partial}{\partial y}\left(z^2\right) + \frac{\partial}{\partial z}(xyz) \\ &= 2x + 0 + xy \\ &= 2x + xy \end{aligned}$$

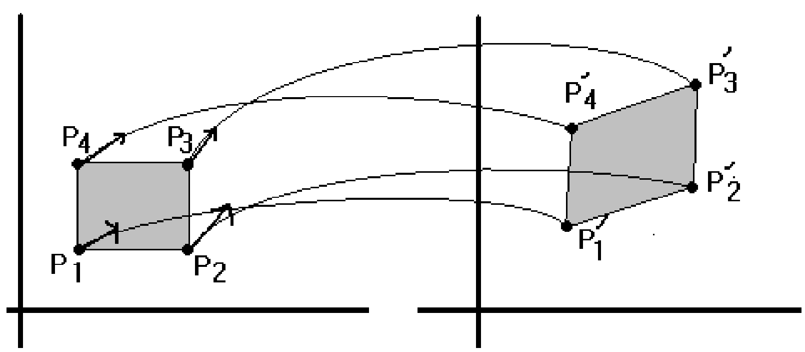
REMARK 14.3. If we imagine a vector field $\mathbf{F}$ to represent the flow of a gas or fluid in $\mathbb{R}^3$ then $\nabla \cdot \mathbf{F}$ represents the rate of expansion of the fluid per unit volume. More precisely,

PROPOSITION 14.4. $\nabla \cdot \mathbf{F}$ *measures the rate at which the volume of a fluid element changes under the flow of the vector field* $\mathbf{F}$*.*

Proof. I'll just sketch the proof for the two-dimensional case. The basic idea underlying the proof is to look at how an infinitesimal rectangle is distorted by the flow of $\mathbf{F}$. Let R be a rectangle in $\mathbb{R}^2$ bounded by the points $\mathbf{P}_1 = (x,y)$, $\mathbf{P}_2 = (x + \Delta x, y)$, $\mathbf{P}_3 = (x + \Delta x, y + \Delta y)$, and $\mathbf{P}_4 = (x, y + \Delta y)$. The area (thought of as a 2-dimensional volume) of this rectangle is obviously $V_0 = \Delta x \Delta y$.

Now let's follow the points $\mathbf{P}_i$ as they move along the flow lines of the vector field $\mathbf{F}$. After a time $\Delta t << 1$ the points $\mathbf{P}_i$ move to new positions

$$
\begin{aligned}
\mathbf{P}_1' &= \mathbf{P}_1 + \Delta t \frac{d\sigma_1}{dt}(0) \\
&= (x,y) + \Delta t \mathbf{F}\,(x,y)) \\
&= (x + F_x(x,y)\Delta t, y + F_y(x,y)\Delta t) \\
\mathbf{P}_2' &= \mathbf{P}_2 + \Delta t \frac{d\sigma_2}{dt}(0) \\
&= (x+\Delta x, y) + \Delta t \mathbf{F}\,(x+\Delta x, y) \\
&= (x + \Delta x + F_x(x+\Delta x, y)\Delta t, y + F_y(x+\Delta x, y)\Delta t) \\
&= \left(x + \Delta x + \left(F_x(x,y) + \frac{\partial F_x}{\partial x}(x,y)\Delta x\right)\Delta t, y + \left(F_y(x,y) + \frac{\partial F_y}{\partial x}(x,y)\Delta x\right)\Delta t\right) \\
\mathbf{P}_3' &= \mathbf{P}_3 + \Delta t \frac{d\sigma_3}{dt}(0) \\
&= (x+\Delta x, y+\Delta y) + \Delta t \mathbf{F}\,(x+\Delta x, y+\Delta y) \\
&= \left(x + \Delta x + \left(F_x(x,y) + \frac{\partial F_x}{\partial x}(x,y)\Delta x + \frac{\partial F_x}{\partial y}\Delta y\right)\Delta t, y + \Delta y + \left(F_y(x,y) + \frac{\partial F_y}{\partial x}(x,y)\Delta x + \frac{\partial F_y}{\partial y}\Delta y\right)\Delta t\right) \\
\mathbf{P}_4' &= \mathbf{P}_4 + \Delta t \frac{d\sigma_4}{dt}(0) \\
&= (x, y+\Delta y) + \Delta t \mathbf{F}\,(x, y+\Delta y) \\
&= \left(x + \left(F_x(x,y) + \frac{\partial F_x}{\partial y}\Delta y\right)\Delta t, y + \Delta y + \left(F_y(x,y) + \frac{\partial F_y}{\partial y}\Delta y\right)\Delta t\right)
\end{aligned}
$$



Let us set consider the positions of the vertices relative to $\mathbf{P}_1$ and $\mathbf{P}_1'$

$$
\begin{aligned}
\mathbf{L}_{12} &= \mathbf{P}_2 - \mathbf{P}_1 = (\Delta x, 0) \\
\mathbf{L}_{13} &= \mathbf{P}_3 - \mathbf{P}_1 = (\Delta x, \Delta y) \\
\mathbf{L}_{14} &= \mathbf{P}_4 - \mathbf{P}_1 = (0, \Delta y)
\end{aligned}
$$

$$\begin{aligned}
\mathbf{L}'_{12} &= \mathbf{P}'_2 - \mathbf{P}'_1 \\
&= \left(\Delta x + \frac{\partial F_x}{\partial x}\Delta x \Delta t, \frac{\partial F_y}{\partial x}\Delta x \Delta t\right) \\
\mathbf{L}'_{13} &= \mathbf{P}'_3 - \mathbf{P}'_1 \\
&= \left(\Delta x + \frac{\partial F_x}{\partial x}\Delta x \Delta t + \frac{\partial F_x}{\partial y}\Delta y \Delta t, \Delta y + \frac{\partial F_y}{\partial x}\Delta x \Delta t + \frac{\partial F_y}{\partial y}\Delta y \Delta t\right) \\
\mathbf{L}'_{14} &= \mathbf{P}_4 - \mathbf{P}'_1 \\
&= \left(\frac{\partial F_x}{\partial y}\Delta y \Delta t, \Delta y + \frac{\partial F_y}{\partial x}\Delta y \Delta t\right)
\end{aligned}$$

Note that $\mathbf{L}'_{13} = \mathbf{L}'_{12} + \mathbf{L}'_{14}$, so the image of the original rectangle under the flow of $\mathbf{F}$ is a parallelogram (for sufficiently small Δt). To compute the area of this parallelogram we use a result from high school geometry: **the area of a parallelogram is equal to the product of the lengths of two adjacent sides and the sine of the angle between them**. In other words,

$$A' = \|\mathbf{L}'_{12}\| \, \|\mathbf{L}'_{14}\| \sin(\theta)$$

where θ is the angle between $\mathbf{L}'_{12}$ and $\mathbf{L}'_{14}$. But the right hand side is also interpretable as the magnitude of the cross product of $\mathbf{L}'_{12}$ and $\mathbf{L}'_{14}$. Thus,

$$A' = \|\mathbf{L}'_{12} \times \mathbf{L}'_{14}\|$$

Now,

$$\begin{aligned}
\mathbf{L}'_{12} \times \mathbf{L}'_{14} &= (\Delta x + \frac{\partial F_x}{\partial x}\Delta x \Delta t, \frac{\partial F_y}{\partial x}\Delta x \Delta t, 0) \times \left(\frac{\partial F_x}{\partial y}\Delta y \Delta t, \Delta y + \frac{\partial F_y}{\partial x}\Delta y \Delta t, 0\right) \\
&= \left(0, 0, \left(\Delta x + \frac{\partial F_x}{\partial x}\Delta x \Delta t\right)\left(\Delta y + \frac{\partial F_y}{\partial x}\Delta y \Delta t\right) - \left(\frac{\partial F_y}{\partial x}\Delta x \Delta t\right)\left(\frac{\partial F_x}{\partial y}\Delta y \Delta t\right)\right)
\end{aligned}$$

Since $\mathbf{L}'_{12} \times \mathbf{L}'_{14}$ has only one non-zero component

$$\begin{aligned}
A' &= \|\mathbf{L}'_{12} \times \mathbf{L}'_{14}\| \\
&= (\mathbf{L}'_{12} \times \mathbf{L}'_{14})_z \\
&= \left(\Delta x + \frac{\partial F_x}{\partial x}\Delta x \Delta t\right)\left(\Delta y + \frac{\partial F_y}{\partial x}\Delta y \Delta t\right) - \left(\frac{\partial F_y}{\partial x}\Delta x \Delta t\right)\left(\frac{\partial F_x}{\partial y}\Delta y \Delta t\right) \\
&= \Delta x \Delta y + \frac{\partial F_x}{\partial x}\Delta x \Delta y \Delta t + \frac{\partial F_y}{\partial x}\Delta x \Delta y \Delta t \\
&+ \frac{\partial F_x}{\partial x}\frac{\partial F_y}{\partial x}\Delta x \Delta y (\Delta t)^2 - \frac{\partial F_y}{\partial x}\frac{\partial F_x}{\partial y}\Delta x \Delta y (\Delta t)^2 \\
&= \Delta x \Delta y \left(1 + \nabla \cdot \mathbf{F} \Delta t + \mathcal{O}\left(\Delta t^2\right)\right)
\end{aligned}$$

So to the first order in Δt the change in area under the flow of $\mathbf{F}$ is

$$\begin{aligned}
A' - A &= \Delta x \Delta y \left(1 + \nabla \cdot \mathbf{F} \Delta t\right) - \Delta x \Delta y \\
&= \Delta x \Delta y \left(\nabla \cdot \mathbf{F}\right) \Delta t \\
&= A \left(\nabla \cdot \mathbf{F}\right) \Delta t
\end{aligned}$$

Thus the rate at which an area A is decreasing or increasing under the flow of $\mathbf{F}$ will be

$$\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \lim_{\Delta t \to 0} \frac{A \left(\nabla \cdot \mathbf{F}\right) \Delta t}{\Delta t} = A \left(\nabla \cdot \mathbf{F}\right)$$

2. The Curl of a Vector Field

Definition 14.5. *Let* $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ *be a differentiable vector field. The* ***curl*** *of* $\mathbf{F}$*, denoted* $\nabla \times \mathbf{F}$*, is the vector-valued function on* $\mathbb{R}^n$ *whose component functions are*

$$(\nabla \times \mathbf{F})_x = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}$$
$$(\nabla \times \mathbf{F})_y = \frac{\partial F_x}{\partial z} - \frac{\partial F_x}{\partial z}$$
$$(\nabla \times \mathbf{F})_x = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$$

3.