LECTURE 13

Vector Fields

Definition 13.1. *A **vector field** on $\mathbb{R}^n$ is a function $\mathbf{F} : A \subset \mathbb{R}^n \to \mathbb{R}^n$ that assignes to each point $\mathbf{x}$ in its domain A an n-dimensional vector $\mathbf{F}(\mathbf{x})$.*

Example 13.2. The gradient ∇f of a function $f : \mathbb{R}^n \to \mathbb{R}$ is a vector field. For

$$\nabla f(\mathbf{x}) = \left(\frac{\partial f}{\partial x_1}(\mathbf{x}), \frac{\partial f}{\partial x_2}(\mathbf{x}), \ldots, \frac{\partial f}{\partial x_n}(\mathbf{x}) \right)$$

is always an n-dimensional vector.

Example 13.3. Suppose we define the gravitational field $\mathbf{G}(\mathbf{x})$ at a point $\mathbf{x} \in \mathbb{R}^3$ as the acceleration $\mathbf{a}$ that a particle of unit mass experiences when released from the point $\mathbf{x}$. A gravitational field is then a function which assigns a 3-dimensional vector to each point $\mathbf{x} \in \mathbb{R}^3$. A gravitational field is thus a vector field.

Example 13.4. Similarly, one can define the electric field $\mathbf{E}(\mathbf{x})$ at the point $\mathbf{x} \in \mathbb{R}^3$ in terms of the acceleration that a charged particle experiences when released from the point $\mathbf{x}$. The electric field is also a vector field.

Example 13.5. Let $\mathbf{V}(\mathbf{x})$ be the vector indicating the direction and speed at which a fluid is flowing at a point $\mathbf{x} \in \mathbb{R}^3$. This is also an example of a vector field.

This last example is particularly important - because it is the basis for much of our intuitive understanding of vector fields.

Definition 13.6. *If $\mathbf{F}$ is a vector field, then a flow line of $\mathbf{F}$ is an path $\mathbf{c}(t)$ such that*

$$\frac{d\mathbf{c}}{dt}(t) = \mathbf{F}\left(\mathbf{c}(t)\right)$$

Remark 13.7. If $\mathbf{V}(\mathbf{x})$ is the vector field corresponding to the flow of a fluid, then a flow line of $\mathbf{V}$ is precisely the trajectory that a small particle would travel if dropped in the fluid.

Example 13.8. Show that

$$\mathbf{c}(t) = (\cos(t), \sin(t), t)$$

is a flow line for the vector field

$$\mathbf{F}(x, y, z) = (-y, x, 1)$$

- We have

  $$\frac{d\mathbf{c}}{dt}(t) = (-\sin(t), \cos(t), 1)$$

  and

  $$\mathbf{F}\left(\mathbf{c}(t)\right) = \mathbf{F}\left(\cos(t), \sin(t), t\right) = (-\cos(t), \sin(t), 1)$$

  Thus,

  $$\frac{d\mathbf{c}}{dt}(t) = \mathbf{F}\left(\mathbf{c}(t)\right)$$

  and so $\mathbf{c}(t)$ is a flow line for $\mathbf{F}$.

EXAMPLE 13.9. Sketch the flow lines for the vector field $\mathbf{F}(x,y) = \left(\frac{y}{2}, -\frac{x}{2}\right)$.

- A flow line is going to be a path $\sigma(t)$ such that
$$\frac{d\sigma}{dt}(t) = \mathbf{F}\left(\sigma(t)\right) = \left(\frac{y(t)}{2}, -\frac{x(t)}{2}\right)$$
or
$$\frac{dx}{dt} = \frac{y(t)}{2} \tag{13.1}$$
$$\frac{dy}{dt} = -\frac{x(t)}{2} \tag{13.2}$$
This is a set of coupled differential equations. To solve this system we can differerentiate the first equation, and then use the second equation on the right hand side, to get a second order ordinary differerential equation for σ_x :
$$\begin{aligned}\frac{d^2x}{dt^2} &= \frac{1}{2}\frac{dy}{dt}\\ &= \frac{1}{2}\left(-\frac{x}{2}\right)\end{aligned}$$
or
$$\frac{d^2x}{dt^2} + \frac{1}{4}x = 0$$
This is a second order linear differential equation with constant coefficients. The general solution of this equation is
$$x(t) = c_1\cos\left(\frac{t}{2}\right) + c_2\sin\left(\frac{t}{2}\right)$$
To find $\sigma_y(t)$ we can now use (13.1) to find
$$y(t) = 2\frac{dx}{dt} = -c_1\sin\left(\frac{t}{2}\right) + c_2\cos\left(\frac{t}{2}\right)$$
Note that
$$\begin{aligned}(x(t))^2 + \left(y(t)^2\right) &= c_1^2\cos^2\left(\frac{t}{2}\right) + 2c_1c_2\cos\left(\frac{t}{2}\right)\sin\left(\frac{t}{2}\right) + c_2\sin^2\left(\frac{t}{2}\right)\\ &\quad + c_1^2\sin^2\left(\frac{t}{2}\right) - 2c_1c_2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right) + c_2\cos^2\left(\frac{t}{2}\right)\\ &= c_1^2\left(\cos^2\left(\frac{t}{2}\right) + \sin^2\left(\frac{t}{2}\right)\right) + c_2^2\left(\sin^2\left(\frac{t}{2}\right) + \cos^2\left(\frac{t}{2}\right)\right)\\ &= c_1^2 + c_2^2\end{aligned}$$
So every point on a such a flow line lies on a circle of radius $R = \sqrt{c_1^2 + c_2^2}$.