LECTURE 12

Arc Length

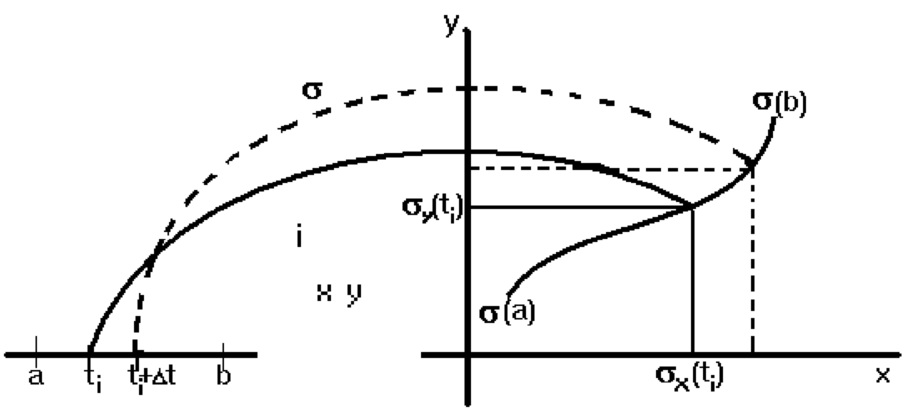
Suppose $\sigma : [a,b] \to \mathbb{R}^2$ is a path in the xy-plane. How does one compute the length of the corresponding curve?

Well suppose we break the interval $[a,b]$ up into n subintervals of length

$$\Delta t = \frac{b-1}{n}$$

It we make Δt small enough (by making n large enough), then the curve between $\sigma(t_i)$ and $\sigma(t_i + \Delta t)$ will be almost a straight line. So if we define d_i to be the distance between the points $\sigma(t_i)$ and $\sigma(t_i + \Delta t)$ in the plane, then

$$\begin{aligned} d_i &= \|\sigma(t_i + \Delta t) - \sigma(t_i)\| \\ &= \sqrt{(\sigma_x(t_i + \Delta t) - \sigma_x(t_i))^2 + (\sigma_y(t_i + \Delta t) - \sigma_y(t_i))^2} \end{aligned}$$



Now

$$\begin{aligned} \sigma_x(t_i + \Delta t) &\approx \sigma_x(t_i) + \frac{d\sigma_x}{dt}(t_i)\Delta t \\ \sigma_y(t_i + \Delta t) &\approx \sigma_y(t_i) + \frac{d\sigma_y}{dt}(t_i)\Delta t \end{aligned}$$

so

$$\begin{aligned} d_i &\approx \sqrt{\left(\frac{d\sigma_x}{dt}(t_i)\Delta t\right)^2 + \left(\frac{d\sigma_y}{dt}(t_i)\Delta t\right)^2} \\ &= \sqrt{\left(\frac{d\sigma_x}{dt}(t_i)\right)^2 + \left(\frac{d\sigma_y}{dt}(t_i)\right)^2}\,\Delta t \\ &= \left\|\frac{d\sigma}{dt}(t_i)\right\| \Delta t \end{aligned}$$

To estimate the total length of the curve

$$\mathbf{c} = \left\{ \mathbf{x} \in \mathbb{R}^2 \mid \mathbf{x} = \sigma(t) \quad , \quad t \in [a,b] \right\}$$

we simply sum the lengths d_i of the straight line approximations to curve between $\sigma(t_i)$ and $\sigma(t_{i+1}) = \sigma(t_i + \Delta t)$.

$$\begin{aligned} L &\approx \sum_{i=1}^{n} \|\sigma(t_i + \Delta t) - \sigma(t_i)\| \\ &= \sum_{i=1}^{n} d_i \\ &= \sum_{i=1}^{n} \left\| \frac{d\sigma}{dt}(t_i) \right\| \Delta t \end{aligned}$$

Recognizing the right hand sum as a Riemann sum, we can take the limit as $\Delta t \to 0$, $n \to \infty$ and replace the summation on the right with a Riemann integral. We thus arrive at the formula

$$L = \int_a^b \left\| \frac{d\sigma}{dt}(t) \right\| dt$$

Remark 12.1. There is nothing in the above derivation that is peculiar to curves in the plane. This formula works just as well for curves in an n-dimensional space. More precisely, if $\sigma : [a,b] \to \mathbb{R}^n$ is a parameterized path in $\mathbb{R}^n$, then the length of the corresponding curve in $\mathbb{R}^n$ is precisely

$$L = \int_a^b \left\| \frac{d\sigma}{dt}(t) \right\| dt$$

It's just that $\left\| \frac{d\sigma}{dt}(t) \right\|$will now be the magnitude of an n-dimensional vector rather than a 2-dimensional vector.

Example 12.2. Use the arc length formula above to compute the length of the circumference of a circle of radius r.

- The first thing we need is a parameterization for a circle of radius r. This should be pretty familar; we take

  $$\sigma(t) : [0, 2\pi] \to \mathbb{R}^2 \quad : \quad t \mapsto (r\cos(t), r\sin(t))$$

  We then have

  $$\frac{d\sigma}{dt}(t) = (-r\sin(t), r\cos(t))$$

  and

  $$\left\| \frac{d\sigma}{dt}(t) \right\| = \sqrt{(-r\sin(t))^2 + (r\cos(t))^2} = \sqrt{r^2\left(\sin^2(t) + \cos^2(t)\right)} = \sqrt{r^2} = r$$

  The arc length formula developed above now yields

  $$L = \int_0^{2\pi} \left\| \frac{d\sigma}{dt}(t) \right\| dt = \int_0^{2\pi} r\,dt = rt\Big|_0^{2\pi} = 2\pi r$$

  Thus, our fancy new-fangled formalism certainly agrees with what we learned in high school.

Theorem 12.3. *If $\sigma_1 : [a,b] \to \mathbb{R}^n$ and $\sigma_2 : [c,d] \to \mathbb{R}^n$ are two one-to-one parametric paths parameterizing the same curve then*

$$\int_c^d \left\| \frac{d\sigma_2}{dt}(t) \right\| dt = \int_a^b \left\| \frac{d\sigma_1}{dt}(t) \right\| dt$$

i.e., the arc-length of the curve is independent of the parameterization.

Proof. Since σ_1 and σ_2 are two one-to-one paths with the same image curve, by the Theorem 9.3 (in Lecture 9) we have a one-to-one mapping h from $[c,d]$ to $[a,b]$ such that

$$\sigma_2 = \sigma_1 \circ h$$

Note that since h is both continuous and one-to-one, endpoints must map to endpoints; and so either $h(c) = a$ and $h(d) = b$; or $h(c) = b$ and $h(d) = a$. Now by the chain rule

$$\frac{d\sigma_2}{dt}(t) = \mathbf{D}\sigma_1|_{h(t)}\,\mathbf{D}h(t) = \left.\frac{d\sigma_1}{dt}\right|_{h(t)} \left.\frac{dh}{dt}\right|_t$$

and so

$$\left\|\frac{d\sigma_2}{dt}(t)\right\| = \left\|\frac{d\sigma_1}{dt}\,(h(t))\,\frac{dh}{dt}(t)\right\| = \left\|\frac{d\sigma_1}{dt}\,(h(t))\right\|\left|\frac{dh}{dt}(t)\right|$$

Thus,

$$\int_c^d \left\|\frac{d\sigma_2}{dt}(t)\right\| dt = \int_c^d \left\|\frac{d\sigma_1}{dt}\,(h(t))\right\|\left|\frac{dh}{dt}(t)\right| dt$$

If we now make a change of variable

$$\begin{aligned} s &= h(t) \\ ds &= \frac{dh}{dt}(t)dt = \pm\left|\frac{dh}{dt}(t)\right| dt \end{aligned}$$

$$\int_c^d \left\|\frac{d\sigma_2}{dt}(t)\right\| dt = \int_c^d \left\|\frac{d\sigma_1}{dt}\,(h(t))\right\|\left|\frac{dh}{dt}(t)\right| dt = \pm\int_{h(c)}^{h(d)} \left\|\frac{d\sigma_1}{dt}(s)\right\| ds$$

Now the $\pm$ sign out front on the right hand side is the sign of $\frac{dh}{dt}$ which depends on whether or not $h(t)$ is an increasing or decreasing function. Note that since h is continuous and one-to-one, the sign of $\frac{dh}{dt}$ is always positive or always negative (if $\frac{dh}{dt}$ flipped a sign, then $h(t)$ would have to double back on itself). Now if $h(t)$ is always increasing then $c < d$ implies $h(c) < h(d)$. Since $h(c)$ and $h(d)$ must also be endpoints for the interval $[a,b]$ we have

$$\frac{dh}{dt} > 0 \quad \Rightarrow \quad h(c) = a \quad , \quad h(d) = b$$

On the other hand, if $h(t)$ is always decreasing then since $c < d$, we must have $h(c) > h(d)$. But since $h(c)$ and $h(d)$ must also be endpoints for the interval $[a,b]$ we must have

$$\frac{dh}{dt} < 0 \quad \Rightarrow \quad h(c) = b \quad , \quad h(d) = a$$

Thus

$$\int_c^d \left\|\frac{d\sigma_2}{dt}(t)\right\| dt = \pm\int_{h(c)}^{h(d)} \left\|\frac{d\sigma_1}{dt}(s)\right\| ds = \left\{\begin{array}{ll} +\int_a^b \left\|\frac{d\sigma_1}{dt}(s)\right\| ds & \text{if } \frac{dh}{dt} > 0 \\ -\int_b^a \left\|\frac{d\sigma_1}{dt}(s)\right\| ds & \text{if } \frac{dh}{dt} < 0 \end{array}\right. = \int_a^b \left\|\frac{d\sigma_1}{dt}(t)\right\| dt$$

$\square$