LECTURE 9

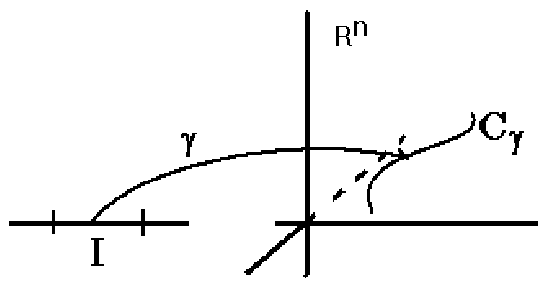
Paths and Curves in $\mathbb{R}^n$

We have used the notion of a path $\gamma : \mathbb{R} \to \mathbb{R}^n$ several times already, I now want formalize this fundamental idea.

DEFINITION 9.1. *A **(parametric) path** in $\mathbb{R}^n$ is a continuous function $\gamma : I \subset \mathbb{R} \to \mathbb{R}^n$ from some interval I on the real line into $\mathbb{R}^n$.*

DEFINITION 9.2. *A **curve** in $\mathbb{R}^n$ is the image of a path in $\mathbb{R}^n$: i.e., the curve corresponding to a path $\gamma : I \subset \mathbb{R} \to \mathbb{R}^n$ is*

$$C_\gamma = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \gamma(t) \text{ for some } t \in I\}$$



Although at this stage, we're careful to distinguish between the function that defines the curve and its image in $\mathbb{R}^n$ we'll no doubt soon lapse into a usage in which the both words *curve* and *path* can mean either a function $\gamma : I \subset \mathbb{R} \to \mathbb{R}^n$ or its image. Nevertheless, the proper interpretation of these words should be clear from the context.

The reason for making a distinction at this junction though is because to a particular curve there can correspond infinitely many parametric paths. To see this note that the image of the path

$$\gamma_1 : [0,\infty] \subset \mathbb{R} \to \mathbb{R}^3 : \gamma_1(t) = (t,t,t)$$

coincides with the images of

$$\gamma_2 : [0,\infty] \subset \mathbb{R} \to \mathbb{R}^3 : \gamma_2(t) = (t^2,t^2,t^2)$$
$$\gamma_3 : [0,\infty] \subset \mathbb{R} \to \mathbb{R}^3 : \gamma_3(t) = (t^3,t^3,t^3)$$

and even that of

$$\gamma_1 : [0,\infty] \subset \mathbb{R} \to \mathbb{R}^3 : \gamma_4(t) = (e^t,e^t,e^t)$$

The following theorem tells us when and how two parametric paths might correspond to the same curve.

THEOREM 9.3. *Suppose $\gamma_1 : I_1 \subset \mathbb{R} \to \mathbb{R}^n$ and $\gamma_2 : I_2 \subset \mathbb{R} \to \mathbb{R}^n$ are both continuous one-to-one maps and correspond to the same curve C: i.e,*

$$\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \gamma_1(t) \text{ for some } t \in I_1\} = C = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \gamma_2(t) \text{ for some } t \in I_2\}$$

Then there exists a one-to-one map h from I_2 to I_1 such that

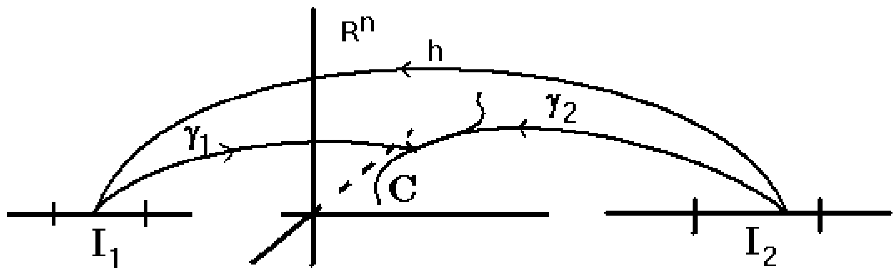
$$\gamma_2 = \gamma_1 \circ h$$

Proof. Since γ_1 is a one-to-one map onto C, its inverse $\gamma_1^{-1} : C \rightarrow I_1$ is well-defined and is also one-to-one. Set

$$h = \gamma_1^{-1} \circ \gamma_2$$

This function clearly maps I_2 onto I_1 and because it is the composition of a pair of one-to-one maps, it is also one-to-one. Finally,

$$\gamma_1 \circ h = \gamma_1 \circ \left(\gamma_1^{-1} \circ \gamma_2\right) = \left(\gamma_1 \circ \gamma_1^{-1}\right) \circ \gamma_2 = \gamma_2$$



REMARK 9.4. If one thinks of the map $h : I_2 \rightarrow I_1$ as a **reparameterization** of the interval I_1, then the theorem says that if you have two one-to-one parameterized paths with the same image curve, then one path is always interpretable as a reparameterization of the other and vice-versa. Lacking a canoical choice of parametric path for a given curve, we say that the parametric path corresponding to a given curve is only determined up to a choice of parameterization. I note that this idea in turn is the basis for the most prominient unified quantum theory (string theory) today.

DEFINITION 9.5. *If $\gamma : I \subset \mathbb{R} \rightarrow \mathbb{R}^n$ is a differentiable path then the* ***tangent vector*** *$\gamma'(t)$ to the path γ at the point $\gamma(t)$ is*

$$\mathbf{D}\gamma(t) = \begin{pmatrix} \frac{d\gamma_1}{dt}(t) \\ \frac{d\gamma_2}{dt}(t) \\ \vdots \\ \frac{d\gamma_n}{dt}(t) \end{pmatrix} \approx \left(\frac{d\gamma_1}{dt}(t), \frac{d\gamma_2}{dt}(t), \ldots, \frac{d\gamma_n}{dt}(t)\right) \in \mathbb{R}^n$$

REMARK 9.6. If we think of a path $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3 : t \mapsto \gamma(t)$ as a function that prescribes the position of a particle at time t then we can regard the corresponding curve C_γ as the *trajectory* of the particle and the tangent vector at $\gamma(t)$ as the *velocity vector* at time t.

DEFINITION 9.7. *If $\gamma(t)$ is a path, and if $\gamma(t_0) \neq \mathbf{0}$, then the equation of the tangent line to the curve C_γ at the point $\gamma(t_0)$ is*

$$\mathbf{x} = \gamma(t_0) + \gamma'(t_0)(t - t_0)$$

EXAMPLE 9.8. Suppose a particle moves along a trajectory described by the function

$$\mathbf{x}(t) = (\cos(2t), \sin(2t), t)$$

What is the velocity of the particle at time $t = 3$.

- The trajectory of the particle turns out to be a helix about the z-axis. The velocity vector at time t is just the tangent vector at time t:

$$\frac{d\mathbf{x}}{dt}(t) = (-2\sin(t), 2\cos(t), 1)$$

  which at time $t = 3$ is

$$\frac{d\mathbf{x}}{dt}(3) = (-2\sin(3), 2\cos(3), 1)$$

EXAMPLE 9.9. Show that the curve prescribed by

$$\mathbf{x}(t) = (2t\cos(t), t\sin(t), -t\sin(t))$$

lies completely in a plane and identify that plane.

- Let's first compute the tangent vector to curve at an arbitrary point $\mathbf{x}(t)$.

  $$\frac{d\mathbf{x}}{dt}(t) = (2\cos(t) - 2t\sin(t), \sin(t) + t\cos(t), -\sin(t) - t\cos(t))$$

  At $t = 0$ we have

  $$\frac{d\mathbf{x}}{dt}(0) = (2, 0, 0)$$

  at $t = \pi/2t$ we have

  $$\frac{d\mathbf{x}}{dt}\left(\frac{\pi}{2}\right) = (-\pi, 1, -1)$$

  In order to find a vector perpendicular to these two vectors we compute their cross product:

  $$\begin{aligned} \mathbf{n} &= (2,0,0) \times (-\pi, 1, -1) \\ &= ((0)(-1) - (0)(1), (0)(-\pi) - (2)(-1), (2)(1) - (0)(-\pi)) \\ &= (0, 2, 2) \end{aligned}$$

  We now show that $\mathbf{n}$ is perpendicular to every tangent vector to the curve $\mathbf{x}(t)$:

  $$\begin{aligned} \mathbf{n} \cdot \frac{d\mathbf{x}}{dt}(t) &= (0,2,2) \cdot (2\cos(t) - 2t\sin(t), \sin(t) + t\cos(t), -\sin(t) - t\cos(t)) \\ &= 0 + 2(\sin(t) + t\cos(t)) + 2(-\sin(t) - t\cos(t)) \\ &= 0 \end{aligned}$$

  Since $\mathbf{n}$ is perpendicular to every tangent vector of $\mathbf{x}(t)$ the corresponding curve never leaves the plane defined by the equation

  $$\begin{aligned} 0 &= \mathbf{n} \cdot (\mathbf{x} - \mathbf{x}(0)) \\ &= (0,2,2) \cdot (x - 0, y - 0, z - 0) \\ &= 2y - 2z \end{aligned}$$