LECTURE 6

Partial Derivatives and Differentiability

1. Partial Derivatives

DEFINITION 6.1. *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be a real-valued function. Then the* $\boldsymbol{i^{th}}$ ***partial derivative*** *is the real-valued function*

$$\begin{aligned}\frac{\partial f}{\partial x_i}(\mathbf{x}) &= \lim_{h\to 0}\frac{f(\mathbf{x}+h\mathbf{e}_i)-f(\mathbf{x})}{h}\\ &= \lim_{h\to 0}\frac{f(x_1,\dots,x_{i-1},x_i+h,x_{i+1},\dots,x_n)-(x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_n)}{h}\end{aligned}$$

REMARK 6.2. Note that limit used in the above definition is just the limit of a function of a single variable. Indeed, fix $\mathbf{x} = (x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)$ and define

$$F(h) = (x_1, \dots, x_{i-1}, h, x_{i+1}, \dots, x_n)$$

Then

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) = \lim_{h\to 0}\frac{F(x_i+h)-F(x_i)}{h} \equiv \left.\frac{dF}{dh}\right|_{h=x_i}$$

This observation in fact tells us precisely how to compute the i^{th} partial derivative of f. We regard all the coordinates except x_i to be fixed (i.e., constants) and then use ordinary calculus to differentiate the resulting function of x_i (regarded as a function of a single variable).

EXAMPLE 6.3. Let $f(x,y) = \cos(x^2y^2)$. Then

$$\begin{aligned}\frac{\partial f}{\partial x} &= -\sin(x^2y^2)(2xy^2)\\ \frac{\partial f}{\partial y} &= -\sin(x^2y^2)(2x^2y)\end{aligned}$$

The ordinary derivative $\frac{df}{dx}$of a function f of a single variable x tells us the rate of change of a function as x increases. Similarly, the partial derivative is interpretable as a rate of change: the partial derivative

$$\frac{\partial f}{\partial x_i}(\mathbf{x})$$

is the rate at which the function f changes as one moves away from the point $\mathbf{x}$ in the direction $\mathbf{e}_i$.

2. Differentiability of Functions of Several Variables

Recall that in the case of a function of a single variable, a function $f(x)$ is differentiable only if it is continuous; but that continuity does not guarantee differentiability. Intuitively, continuity of $f(x)$ requires that its graph be a continuous curve; and differentiability requires also that there is always a unique tangent vector to the graph of $f(x)$. In other words, a function $f(x)$ is differentiable if and only if its graph is a smooth continuous curve with no sharp corners (a sharp corner would be a place where there would be two possible tangent vectors).

If we try to extend this graphical picture of differentiability to functions of two or more variables, it would be natural to think of a differentiable function of several variables as one whose graph is a smooth continuous surface, with no sharp peaks or folds. Because for such a surface it would always be possible to associate a unique tangent plane at a given point.

However, "differentiability" in this sense turns out to be a much stronger condition than the mere existence of partial derivatives. For the existence of a partial derivatives at a point $\mathbf{x}_0$ requires only a smooth approach to the point $f(\mathbf{x}_0)$ along the direction of the coordinate axes. We have seen examples of functions that are discontinuous even though

$$\lim_{x\to 0} f(x,0) = \lim_{y\to 0} f(0,y)$$

both exist. For example the function

$$f(x,y) = \frac{(x-y)^2}{x^2+y^2}$$

has this property, and in fact, both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous functions at the point $(0,0)$.

With this sort of phenomenon in mind we give the following definition of differentiability.

Definition 6.4. *We say that a function* $f : \mathbb{R}^2 \to \mathbb{R}$ *of two variables* x *and* y *is* ***differentiable*** *at* (x_0, y_0) *if*

1. *Both* $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ *exist at the point* (x_0, y_0).
2. $$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - \left[\frac{\partial f}{\partial x}(x_0,y_0)\right](x-x_0) - \left[\frac{\partial f}{\partial y}(x_0,y_0)\right](y-y_0)}{\left|(x-x_0)^2 + (y-y_0)^2\right|^{\frac{1}{2}}} = 0$$

Remark 6.5. The limit condition simply means that

$$F(x,y) = f(x_0,y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} (x-x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} (y-y_0)$$

is a good approximation to $f(x,y)$ near the point (x_0,y_0). To make contact with our graphical interpretation of differentiability, we simply note that the graph of $F(x,y)$ is a plane (it is linear in the variables x and y). To make this completely obvious, recall that the solution set of any equation of form

$$Ax + By + Cz = D$$

is a plane in $\mathbb{R}^3$; so taking

$$\begin{aligned}
A &= -\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \\
B &= -\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \\
C &= 1 \\
D &= f(x_0,y_0) - \left[\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}\right] x_0 - \left[\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}\right] y_0
\end{aligned}$$

we see that the equation of the graph of $F(x,y)$

$$z = F(x,y) = -Ax - By + D$$

is a plane. Thus the limit condition is saying that the graph of $f(x,y)$ coincides with the graph of a plane as (x,y) approaches the point (x_0,y_0). This observation motivates the following definition.

DEFINITION 6.6. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function that is differentiable at the point $\mathbf{x}_0 = (x_0, y_0)$. Then the plane in $\mathbb{R}^3$ defined by the equation*

$$z = f(x_0, y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} (x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} (y - y_0)$$

*is called the **plane tangent to the graph of** f at $\mathbf{x}_0$.*

In order to generalize these definitions to the case of a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ it suffices to simply generalize our notation.

DEFINITION 6.7. *If f is any function from $\mathbb{R}^n$ to $\mathbb{R}$ we define the **gradient** ∇f of f to be the function from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by*

$$\nabla f(\mathbf{x}) = \left(\frac{\partial f}{\partial x_1}(\mathbf{x}), \frac{\partial f}{\partial x_2}(\mathbf{x}), \dots , \frac{\partial f}{\partial x_n}(\mathbf{x}) \right)$$

Of course, for this definition to make sense all the partial derivatives $\frac{\partial f}{\partial x_1}(\mathbf{x}), , \dots , \frac{\partial f}{\partial x_n}(\mathbf{x})$ must exist.

DEFINITION 6.8. *Let U be an open set in $\mathbb{R}^n$ and let f be a function from U to $\mathbb{R}$. We say that f is **differentiable** at a point $\mathbf{x}_0 \in U$ if the partial derivatives*

$$\frac{\partial f}{\partial x_i}(\mathbf{x}_0) = \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\mathbf{e}_i) - f(\mathbf{x}_0)}{h}$$

all exist and if

$$0 = \lim_{\mathbf{x} \to \mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{\|\mathbf{x} - \mathbf{x}_0\|}$$

We can also extend our notion of differentiability to functions from $\mathbb{R}^n$ to $\mathbb{R}^m$.

DEFINITION 6.9. *Let U be an open set in $\mathbb{R}^n$ and let f be a function from U to $\mathbb{R}$. We say that f is **differentiable** at a point $\mathbf{x}_0 \in U$ if the partial derivatives*

$$\frac{\partial f_j}{\partial x_i}(\mathbf{x}_0) = \lim_{h \to 0} \frac{f_j(\mathbf{x}_0 + h\mathbf{e}_i) - f_j(\mathbf{x}_0)}{h}$$

all exist and

$$0 = \lim_{\mathbf{x} \to \mathbf{x}_0} \frac{\left\| f(\mathbf{x}) - f(\mathbf{x}_0) - \sum_{j=0}^n \nabla f_j(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)\mathbf{e}_j \right\|}{\|\mathbf{x} - \mathbf{x}_0\|}$$

EXAMPLE 6.10. Let $f(x, y) = (xy, x + y)$. Then

$$T(x, y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix}$$

and so for f to be differentiable at $(1, 0)$ we require

$$\begin{aligned} 0 &= \lim_{\mathbf{x} \to \mathbf{x}_0} \frac{\left\| f(\mathbf{x}) - f(\mathbf{x}_0) - \sum_{j=0}^n \nabla f_j(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)\mathbf{e}_j \right\|}{\|\mathbf{x} - \mathbf{x}_0\|} \\ &= \lim_{\mathbf{x} \to \mathbf{x}_0} \frac{\| f(\mathbf{x}) - f(\mathbf{x}_0) - (\nabla f_1(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0), 0) - (0, \nabla f_2(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)) \|}{\|\mathbf{x} - \mathbf{x}_0\|} \end{aligned}$$

Now

$$\begin{aligned}
f(\mathbf{x}) &= (xy, x+y) \\
f(\mathbf{x}_0) &= f(1,0) = (0,1) \\
\nabla f_1 &= \left(\frac{\partial}{\partial x}(xy), \frac{\partial}{\partial y}(xy)\right) = (y,x) \quad \Rightarrow \quad \nabla f_1(\mathbf{x}_0) = \nabla f_1(1,0) = (0,1) \\
\nabla f_2 &= \left(\frac{\partial}{\partial x}(x+y), \frac{\partial}{\partial y}(x+y)\right) = (1,1) \quad \Rightarrow \quad \nabla f_2(\mathbf{x}_0) = \nabla f_2(1,0) = (1,1) \\
\nabla f_1(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0) &= (0,1)\cdot(x-1, y-0) = y \\
\nabla f_2(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0) &= (1,1)\cdot(x-1, y-0) = x-1+y
\end{aligned}$$

Hence the limit condition becomes

$$\begin{aligned}
0 &= \lim_{(x,y)\to(1,0)} \frac{\|(xy, x+y) - (0,1) - (y,0) - (0, x-1+y)\|}{\|(x-1, y\|} \\
&= \lim_{(x,y)\to(1,0)} \frac{\|(xy - 0 - y - 0, x+y-1-0-x+1-y)\|}{\|(x-1, y\|} \\
&= \lim_{(x,y)\to(1,0)} \frac{\|(xy-y, 0)\|}{\|(x-1, y\|} \\
&= \lim_{(x,y)\to(1,0)} \frac{\sqrt{(xy-y)^2}}{\sqrt{(x-1)^2+y^2}}
\end{aligned}$$

Unfortunately as we have seen in the examples we discussed in the lecture on limits and continuity, it is not so easy to see whether or not this limit exists. Luckily we have a theorem at our disposal that makes it a lot easier to decide questions of differentiability.

THEOREM 6.11. *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$. Suppose that the derivatives $\frac{\partial f_j}{\partial x_i}$ all exist and are continuous on a neighborhood of a point $\mathbf{x}$ in U. Then f is differentiable at $\mathbf{x}$.*