Limits of Real-Valued Functions

1. Topology of \( \mathbb{R}^n \)

Fundamental to an understanding of the functions of a single variable is the notion of an open interval \((a, b) = \{x \in \mathbb{R} \mid a < x < b\}\). The first step in developing a calculus for functions of several variables is to develop a higher dimensional analog of the open interval. For, just as in the case of a function of a single variable, the limit of a function \( f : \mathbb{R}^n \to \mathbb{R} \), say at a point \( p \in \mathbb{R}^n \) will depend not so much on the value of the \( f \) at \( p \), but rather on the values of \( f \) at points “close” to \( p \). This is why we must first explain what it means to be “close” to a given point.

**Definition 5.1.** An open ball of radius \( r \) about a point \( p_0 \) in \( \mathbb{R}^n \) is the set of points \( B_r(p_0) \) defined by

\[
B_r(p_0) = \{ p \in \mathbb{R}^n \mid ||p - p_0|| < r \}
\]

(i.e. the set of all points within distance \( r \) of the point \( p_0 \)).

**Remark 5.2.** We shall often call an open ball about a point \( p_0 \) in \( \mathbb{R}^n \) a neighborhood of \( p_0 \).

**Definition 5.3.** Let \( U \) be a subset of \( \mathbb{R}^n \). We say that \( U \) is a open subset if for every point \( p_0 \in U \) there exists an open ball of radius \( r > 0 \) about \( p_0 \) lying completely within \( U \).

**Remark 5.4.** This is the analog of an open interval on the real line. The main ideas being that such a set \( U \) does not include its boundary, and given any point in the set there’s always another point even closer to the boundary of \( U \). Below we make our notion of boundary a little more precise.

**Definition 5.5.** Let \( U \) be a subset of \( \mathbb{R}^n \). A point \( p_0 \in \mathbb{R}^n \) is called a boundary point of \( U \) if every open ball \( B_r(p_0) \) about \( p_0 \) contains at least one point in \( U \) and one point not in \( U \).

Note that this definition does not imply that a boundary point of \( U \) is necessarily a point in \( U \). We shall denote by \( \partial U \) the set of boundary points of \( U \).

**Theorem 5.6.** For each point \( p_0 \in \mathbb{R}^n \), and any \( r > 0 \) the open ball \( B_r(p_0) \) is an open set in \( \mathbb{R}^n \).

**proof.** Let \( p_1 \in B_r(p_0) \). Then \( ||p_1 - p_0|| < r \). According to the definition of an open set we need to demonstrate that there is an open ball about the point \( p_1 \) that is contained entirely within \( B_r(p_0) \). Set

\[
s = r - ||p - p_0||
\]

so that

\[
r = s + ||p - p_0||
\]

Then for any point \( p \in B_s(p_1) \) we have

\[
||p - p_1|| < s
\]

But then

\[
||p - p_0|| = ||(p - p_1) + (p_1 - p_0)||
\leq ||p - p_1|| + ||p_1 - p_0|| \quad \text{(by the Triangle Inequality)}
< s + ||p_1 - p_0||
< r
\]
2. Limits of Real-Valued Functions

We introduced the notion of open balls about a point \( p \) so that we could obtain a precise way of identifying the points that are neighboring \( p_0 \). We can now introduce the notion of limit.

**Remark 5.7.** Definition 5.8. Let \( S \) be an open subset of \( \mathbb{R}^n \) and let \( x_0 \) be a point in \( S \) or the boundary of \( S \). Let \( f \) be a function from \( S \) to \( \mathbb{R}^m \).

1. If \( N \) is a neighborhood of a point \( y_0 \in \mathbb{R}^m \), we say that \( f \) is **eventually** in \( N \) as \( x \) approaches \( x_0 \) if there exists a neighborhood \( U \) of \( x_0 \) in \( S \) such that
   \[
   x \in U \setminus \{ x \neq x_0, x \in \mathbb{R} \} \implies f(x) \in N.
   \]

2. We say that
   \[
   \lim_{x \to x_0} f(x) = y_0
   \]
   or
   \[
   f(x) \to y_0 \text{ as } x \to x_0
   \]
   if, given any neighborhood \( N \) of \( y_0 \) there is eventually in \( N \) as \( x \) approaches \( x_0 \).

**Remark 5.9.** The reason why we don’t simply require \( x \in U \implies f(x) \in N \) is that this condition by itself is too restrictive. For unless we also stipulate that \( x \neq x_0 \) we can not take limits of (derivative-like) functions of the form
   \[
   f(x) = \frac{g(x) - g(x_0)}{\|x - x_0\|}
   \]
   since the right hand side of (5.1) is undefined when \( x = x_0 \). For the same reason we also require \( x \) to be in the domain \( A \) of the function \( f \) (i.e., \( U \) of a point near the boundary of \( A \) could very well contain points which are not in \( A \); i.e., points where the function \( f \) is undefined.)

**Example 5.10.** Show that
   \[
   \lim_{x \to x_0} x = x_0
   \]

- Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be the function mapping a point \( x \) to itself and let \( N \) be some ball of radius \( r \) about the point \( f(x_0) = x_0 \). To show that \( f \) is eventually in \( N \) we need to find a neighborhood \( U \) of point \( x_0 \) (regarded as a point in the domain of \( f \)) such that if \( x \in U \) then \( f(x) \in N \). Well, that’s easy
enough, simply take \( U = N \); then if \( x \in U \), \( f(x) = x \) is guaranteed to be in \( N \). So \( f \) is eventually in \( N \) as \( x \) approaches \( x_0 \). Since our choice of the neighborhood \( N \) is arbitrary, we can conclude that

\[
\lim_{x \to x_0} x = x_0
\]

**Theorem 5.11.** (Properties of Limits): Let \( f, g \) be any functions from a subset \( A \) of \( \mathbb{R}^n \) to \( \mathbb{R}^m \).

1. If \( \lim_{x \to x_0} f(x) = b_1 \) and \( \lim_{x \to x_0} f(x) = b_2 \) then \( b_1 = b_2 \). (Limits are unique.)
2. If \( \lim_{x \to x_0} f(x) = b \), then \( \lim_{x \to x_0} (cf(x)) = cb \).
3. If \( \lim_{x \to x_0} f(x) = b_1 \) and \( \lim_{x \to x_0} g(x) = b_2 \), then \( \lim_{x \to x_0} (f(x) + g(x)) = b_1 + b_2 \).
4. If \( m = 1 \) and \( \lim_{x \to x_0} f(x) = b_1 \) and \( \lim_{x \to x_0} g(x) = b_2 \), then \( \lim_{x \to x_0} (f(x)g(x)) = b_1b_2 \).
5. If \( m = 1 \) and \( \lim_{x \to x_0} f(x) = b \neq 0 \), then \( \lim_{x \to x_0} (1/f(x)) = 1/b \).
6. If \( f(x) = (f_1(x), f_2(x), \ldots, f_n(x)) \) where \( f_1 : A \to \mathbb{R}, \ldots, f_m : A \to \mathbb{R} \) are the components functions of \( f \), then

\[
\lim_{x \to x_0} f(x) = b = (b_1, \ldots, b_m)
\]

if and only if

\[
\lim_{x \to x_0} f_i(x) = b_i
\]

for each \( i = 1, \ldots, m \).

**Theorem 5.12.** (Criterion for a limit to exist.). Suppose \( f : \mathbb{R}^n \to \mathbb{R} \) is a real-valued function and

\[
\lim_{x \to x_0} f(x) = L
\]

exists, then if \( \gamma : \mathbb{R} \to \mathbb{R}^n \) is any smooth curve in \( \mathbb{R}^n \) such that \( \gamma(0) = x_0 \), then

\[
\lim_{t \to 0} f(\gamma(t)) = L
\]

This theorem tells us that if \( f \) is a function of several variables and a limit \( \lim_{x \to x_0} f(x) \) exists then we can compute the limit by choosing a path \( \gamma(t) \) passing through the limit point \( x_0 \) and calculating the limit of a function of a single variable \( (F(t) = f(\gamma(t))) \). However, if a limit

\[
\lim_{x \to x_0} f(x)
\]

does not exist, this theorem does not necessarily help us see the limit does not exist.

**Example 5.13.** Consider the function

\[
f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}
\]

There is some hope that

\[
\lim_{(x, y) \to (0, 0)} f(x, y)
\]
exists since the only place where the denominator vanishes is the point \((0,0)\); but there the numerator vanishes as well. Let’s assume the limit exists and calculate by considering the path 

\[ \gamma_1 : t \mapsto (t,0) \]

along the \(x\)-axis. We then have

\[ f(\gamma_1(t)) = \frac{t^2 - 0^2}{t^2 + 0^2} = 1 \]

and so if the limit exists

\[ \lim_{x \to x_0} f(x) = \lim_{t \to 0} f(\gamma_1(t)) = \lim_{t \to 0} 1 = 1 \]

However, consider instead a path along the \(y\)-axis 

\[ \gamma_2 : t \mapsto (0,t) \]

we have

\[ f(\gamma_2(t)) = \frac{0^2 - t^2}{0^2 + t^2} = 1 \]

and so by the preceding theorem if the limit \(\lim_{x \to x_0} f(x)\) exists we must also have

\[ \lim_{x \to x_0} f(x) = \lim_{t \to 0} f(\gamma_2(t)) = \lim_{t \to 0} -1 = -1 \]

But this contradicts the calculation of the limit using the path \(\gamma_1(t)\). We conclude that the limit \(\lim_{x \to x_0} f(x)\) does not exist.

**Example 5.14.** Consider the function

\[ f(x,y) = \frac{xy^2}{x^2 + y^4} \]

Let’s try to be a little more clever this time and try to compute the limit

\[ \lim_{x \to x_0} f(x) \]

by considering a whole family of straight lines passing through the origin. Set

\[ \gamma_m(t) = (t,mt) \]

For any given \(m\) this will correspond to a straight line passing through \((0,0)\) with slope \(m\). We now calculate the limit of the values of \(f\) along such a path

\[ \lim_{t \to 0} f(\gamma_m(t)) = \lim_{t \to 0} \frac{t(mt)^2}{(t)^2 + (mt)^4} = \lim_{t \to 0} \frac{tm^2}{1 + m^4t^2} = 0 \]

since the denominator tends to 1 as \(t \to 0\) while the numerator goes to zero. We can conclude that if we approach the origin along any line the the limit of the values of \(f\) is always 0. Is this sufficient to conclude

\[ \lim_{x \to x_0} \frac{xy^2}{x^2 + y^4} = 0 \]

Let’s consider one other curve

\[ \gamma : t \mapsto (t^2,t) \]

This will be a (sideways) parabola that passes through the origin at \(t = 0\). Calculating the limit of the values of \(f\) along this curve yields

\[ \lim_{t \to 0} f(\gamma(t)) = \lim_{t \to 0} \frac{(t^2)(t^2)}{(t^2)^2 + (t)^4} = \lim_{t \to 0} \frac{1}{2} = \frac{1}{2} \neq 0 \]

Since the limit along this parabolic curve does not agree with the limits we obtained by looking at straight lines we must conclude that the limit

\[ \lim_{x \to x_0} f(x) \]

does not exist.
Remark 5.15. The moral of these two examples is that while it can be helpful to look at the values of a function along a curve in order to compute a limit, it is in general not sufficient to use these values to determine if the limit actually exists.

3. Continuous Functions

Definition 5.16. Let \( f : A \subset \mathbb{R}^n \to \mathbb{R}^m \) be a function with domain \( A \). Let \( x_0 \in A \). We say that \( f \) is **continuous at** \( x_0 \) if

1. \( \lim_{x \to x_0} f(x) \) exists
2. \( \lim_{x \to x_0} f(x) = f(x_0) \).

We say that the function \( f \) is **continuous** if \( f \) is continuous at each point \( x \) in its domain.

Remark 5.17. Recall that for a function \( f(x) \) of a single variable is continuous if its graph is an unbroken curve. Similarly, a function \( f(x) \) of several variables is continuous if it is an unbroken surface (i.e., a surface without holes or tears).

Let me conclude this lecture by stating some fundamental properties of continuous functions.

Theorem 5.18. Suppose \( f : A \subset \mathbb{R}^n \to \mathbb{R}^m \) and \( g : A \subset \mathbb{R}^n \to \mathbb{R}^m \) are continuous functions. Then so are

1. \( cf(x) \), where \( c \) is any constant.
2. \( f(x) + g(x) \)
3. \( f(x) \cdot g(x) \)

Theorem 5.19. Suppose \( f : \mathbb{R}^n \to \mathbb{R}^m \) is continuous and \( g : \mathbb{R}^m \to \mathbb{R}^p \) is continuous then so is the composed function \( g \circ f : \mathbb{R}^n \to \mathbb{R}^p \).

Theorem 5.20. Suppose \( f : A \subset \mathbb{R}^n \to \mathbb{R} \) is continuous and nowhere zero on \( A \). Then the quotient \( 1/f(x) \) is continuous.

Theorem 5.21. Suppose \( f : A \subset \mathbb{R}^n \to \mathbb{R}^m \) and

\[
    f(x) = (f_1(x), f_2(x), \ldots, f_m(x))
\]

Then \( f \) is continuous if and only if each \( f_i : A \subset \mathbb{R}^n \to \mathbb{R} \)

is continuous.