LECTURE 5

Limits of Real-Valued Functions

1. Topology of $\mathbb{R}^n$

Fundamental to an understanding of the functions of single variable is the notion of an open interval $(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$. The first step in developing a calculus for functions of several variables is to develop a higher dimensional analog of the open interval. For, just as in the case of a function of a single variable, the limit of a function $f : \mathbb{R}^n \to \mathbb{R}$, say at a point $\mathbf{p} \in \mathbb{R}^n$ will depend not so much on the value of the f at $\mathbf{p}$, but rather on the values of f at points "close" to $\mathbf{p}$. This is why we must first explain what it means to be "close" to a given point.

Definition 5.1. *An open ball of radius r about a point $\mathbf{p}_0$ in $\mathbb{R}^n$ is the set of points $B_r(\mathbf{p}_0)$ defined by*

$$B_r(\mathbf{p}_0) = \{\mathbf{p} \in \mathbb{R}^n \mid \|\mathbf{p} - \mathbf{p}_0\| < r\}$$

(i.e. the set of all points within distance r of the point $\mathbf{p}_0$).

Remark 5.2. We shall often call an open ball about a point $\mathbf{p}_0$ in $\mathbb{R}^n$ a **neighborhood** of $\mathbf{p}_0$.

Definition 5.3. *Let U be a subset of $\mathbb{R}^n$. We say that U is a* ***open subset*** *if for every point $\mathbf{p}_0 \in U$ there exists an open ball of radius $r > 0$ about $\mathbf{p}_0$ lying completely within U.*

Remark 5.4. This is the analog of an open interval on the real line. The main ideas being that such a set U does not include its boundary, and given any point in the set there's always another point even closer to the boundar of U. Below we make our notion of boundary a little more precise.

Definition 5.5. *Let U be a subset of $\mathbb{R}^n$. A point $\mathbf{p}_0 \in \mathbb{R}^n$ is called a boundary point of U if every open ball $B_r(\mathbf{p}_0)$ about $\mathbf{p}_0$ contains at least one point in U and one point not in U.*

Note that this definition does not imply that a boundary point of U is necessarily a point in U. We shall denote by ∂U the set of boundary points of U.

Theorem 5.6. *For each point $\mathbf{p}_0 \in \mathbb{R}^n$, and any $r > 0$ the open ball $B_r(\mathbf{p}_0)$ is an open set in $\mathbb{R}^n$.*

proof. Let $\mathbf{p}_1 \in B_r(\mathbf{p}_0)$. Then $\|\mathbf{p}_1 - \mathbf{p}_0\| < r$. According to the definition of an open set we need to demonstrate that there is an open ball about the point $\mathbf{p}_1$ that is contained entirely within $B_r(\mathbf{p}_0)$. Set

$$s = r - \|\mathbf{p} - \mathbf{p}_0\|$$

so that

$$r = s + \|\mathbf{p} - \mathbf{p}_0\|$$

Then for any point $\mathbf{p} \in B_s(\mathbf{p}_1)$ we have

$$\|\mathbf{p} - \mathbf{p}_1\| < s$$

But then

$$\begin{aligned}
\|\mathbf{p} - \mathbf{p}_0\| &= \|(\mathbf{p} - \mathbf{p}_1) + (\mathbf{p}_1 - \mathbf{p}_0)\| \\
&\leq \|\mathbf{p} - \mathbf{p}_1\| + \|\mathbf{p}_1 - \mathbf{p}_0\| \quad \text{(by the Triangle Inequality)} \\
&< s + \|\mathbf{p}_1 - \mathbf{p}_0\| \\
&< r
\end{aligned}$$

Hence $\mathbf{p}$ also lies in $B_r(\mathbf{p}_0)$.

2. Limits of Real-Valued Functions

We introduced the notion of open balls about a point $\mathbf{p}$ so that we could obtain a precise way of identifying the points that are neighboring $\mathbf{p}_0$. We can now introduce the notion of limit.

REMARK 5.7. DEFINITION 5.8. *Let S be an open subset of $\mathbb{R}^n$ and let $\mathbf{x}_0$ be a point in S or the boundary of S. Let f be a function from S to $\mathbb{R}^m$.*

1. *If N is a neighborhood of a point $\mathbf{y}_0 \in \mathbb{R}^m$, we say that f is **eventually** in N as $\mathbf{x}$ approaches $\mathbf{x}_0$ if there exists a neighborhood U of $\mathbf{x}_0$ in S such that*

$$\mathbf{x} \in U \ , \ \mathbf{x} \neq \mathbf{x}_0 \ , \ \mathbf{x} \in A \quad \Rightarrow \quad f(\mathbf{x}) \in N \ . \tag{5.1}$$

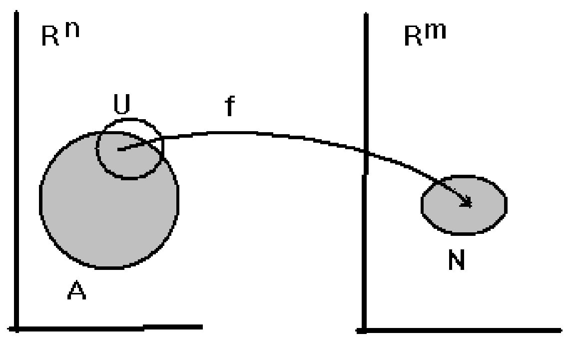
2. *We say that*

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \mathbf{y}_0$$

or

$$f(\mathbf{x}) \to \mathbf{y}_0 \ \textit{as} \ \mathbf{x} \to \mathbf{x}_0$$

*if, given **any** neighborhood N of $\mathbf{y}_0$ there f is eventually in N as $\mathbf{x}$ approaches $\mathbf{x}_0$.*



REMARK 5.9. The reason why we don't simply require $\mathbf{x} \in U \quad \Rightarrow \quad f(\mathbf{x}) \in N$ is that this condition by itself is **too** restrictive. For unless we also stipulate that $\mathbf{x} \neq \mathbf{x}_0$ we can not take limits of (derivative-like) functions of the

$$f(\mathbf{x}) = \frac{g(\mathbf{x}) - g(\mathbf{x}_0)}{\|\mathbf{x} - \mathbf{x}_0\|}$$

since the right hand side of (5.1) is undefined when $\mathbf{x} = \mathbf{x}_0$. For the same reason we also require $\mathbf{x}$ to be in the domain A of the function f (a neighborhood U of a point near the boundary of A could very well contain points which are not in A; i.e., points where the function f is undefined.)

EXAMPLE 5.10. Show that

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{x} = \mathbf{x}_0$$

- Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the function mapping a point $\mathbf{x}$ to itself and let N be some ball of radius r about the point $f(\mathbf{x}_0) = \mathbf{x}_0$. To show that f is eventually in N we need to find a neighborhood U of point $\mathbf{x}_0$ (regarded as a point in the domain of f) such that if $\mathbf{x} \in U$ then $f(\mathbf{x}) \in N$. Well, that's easy enough, simply take $U = N$; then if $\mathbf{x} \in U$, $f(\mathbf{x}) = \mathbf{x}$ is guaranteed to be in N. So f is *eventuall in N as $\mathbf{x}$ approaches $\mathbf{x}_0$*. Since our choice of the neighborhood N is arbitrary, we can conclude that

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{x} = \mathbf{x}_0$$

THEOREM 5.11. ***(Properties of Limits)**: Let f, g be any functions from a subset A of $\mathbb{R}^n$ to R^m.*

1. *If $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}_1$ and $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}_2$ then $\mathbf{b}_1 = \mathbf{b}_2$. (Limits are unique.)*
2. *If $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}$, then $\lim_{\mathbf{x}\to\mathbf{x}_0} (cf(\mathbf{x})) = c\mathbf{b}$.*
3. *If $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}_1$ and $\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x}) = \mathbf{b}_2$, then $\lim_{\mathbf{x}\to\mathbf{x}_0} (f(\mathbf{x}) + g(\mathbf{x})) = \mathbf{b}_1 + \mathbf{b}_2$.*
4. *If $m = 1$ and $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = b_1$ and $\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x}) = b_2$, then $\lim_{\mathbf{x}\to\mathbf{x}_0} (f(\mathbf{x})g(\mathbf{x})) = b_1 b_2$.*
5. *If $m = 1$ and $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = b \neq \mathbf{0}$, then $\lim_{\mathbf{x}\to\mathbf{x}_0} (1/f(\mathbf{x})) = 1/b$.*
6. *If $f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))$ where $f_1 : A \to \mathbb{R}$, $\ldots$, $f_m : A \to \mathbb{R}$ are the components functions of f, then*

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b} = (b_1, \ldots, b_m)$$

if and only if

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f_i(\mathbf{x}) = b_i$$

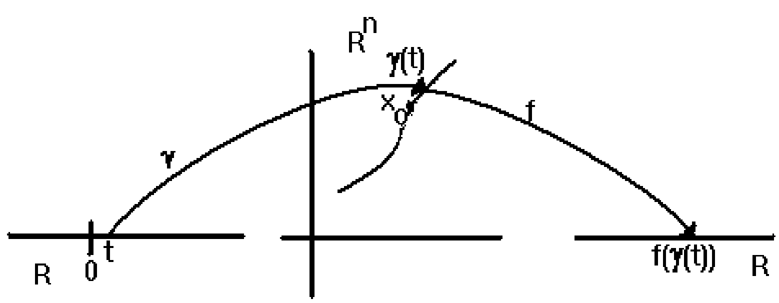
for each $i = 1, \ldots, m$.

THEOREM 5.12. ***(Criterion for a limit to exist.)**. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a real-valued function and*

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = L$$

exists, then if $\gamma : \mathbb{R} \to \mathbb{R}^n$ is any smooth curve in $\mathbb{R}^n$ such that $\gamma(0) = \mathbf{x}_0$, then

$$\lim_{t\to 0} f(\boldsymbol{\gamma}(t)) = L$$



This theorem tells us that if f is a function of several variables and a limit $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})$ exists then we can compute the limit by choosing a path $\gamma(t)$ passing through the limit point $\mathbf{x}_0$ and calculating the limit of a function of a single variable $(F(t) = f(\gamma(t)))$. However, if a limit

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})$$

does not exist, this theorem does not necessarily help us see the limit does not exist.

EXAMPLE 5.13. Consider the function

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

There is some hope that

$$\lim_{(x,y)\to(0,0)} f(x, y)$$

exists since the only place where the denominator vanishes is the point $(0,0)$; but there the numerator vanishes as well. Let's assume the limit exists and calculate by considering the path

$$\gamma_1 : t \mapsto (t,0)$$

along the x-axis. We then have

$$f(\gamma_1(t)) = \frac{t^2 - 0^2}{t^2 + 0^2} = 1$$

and so if the limit exists

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \lim_{t \to 0} f(\gamma_1(t)) = \lim_{t \to 0} 1 = 1$$

However, consider instead a path along the y-axis

$$\gamma_2 : t \mapsto (0,t)$$

we have

$$f\left(\gamma_2(t)\right) = \frac{0^2 - t^2}{0^2 + t^2} = 1$$

and so by the preceding theorem if the limit $\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x})$ exists we must also have

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \lim_{t \to 0} f(\gamma_2(t)) = \lim_{t \to 0} -1 = -1$$

But this contradicts the calculation of the limit using the path $\gamma_1(t)$. We conclude that the limit $\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x})$ does not exist.

Example 5.14. Consider the function

$$f(x,y) = \frac{xy^2}{x^2 + y^4}$$

Let's try to be a little more clever this time and try to compute the limit

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x})$$

by considering a whole family of straight lines passing through the origin. Set

$$\gamma_m(t) = (t, mt)$$

For any given m this will correspond to a straight line passing through $(0,0)$ with slope m. We now calculate the limit of the values of f along such a path

$$\lim_{t \to 0} f\left(\gamma_m(t)\right) = \lim_{t \to 0} \frac{t(mt)^2}{(t)^2 + (mt)^4} = \lim_{t \to 0} \frac{tm^2}{1 + m^4 t^2} = 0$$

since the denominator tends to 1 as $t \to 0$ while the numerator goes to zero. We can conclude that if we approach the origin along any line the the limit of the values of f is always 0. Is this sufficient to conclude

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{xy^2}{x^2 + y^4} = 0 \quad ?$$

Let's consider one other curve

$$\gamma : t \mapsto (t^2, t)$$

This will be a (sideways) parabola that passes through the origin at $t = 0$. Calculating the limit of the values of f along this curve yields

$$\lim_{t \to 0} f(\gamma(t)) = \lim_{t \to 0} \frac{(t^2)(t^2)}{(t^2)^2 + (t)^4} = \lim_{t \to 0} \frac{1}{2} = \frac{1}{2} \neq 0$$

Since the limit along this parabolic curve does not agree with the limits we obtained by looking at straight lines we must conclude that the limit

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x})$$

does not exist.

REMARK 5.15. The moral of these two examples is that while it can be helpful to look at the values of a function along a curve in order to compute a limit, it is in general not sufficient to use these values to determine if the limit actually exists.

3. Continuous Functions

DEFINITION 5.16. *Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ be a function with domain A. Let $\mathbf{x}_0 \in A$. We say that f is* ***continuous at*** $\mathbf{x}_0$ *if*

1. $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})$ *exists*
2. $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0)$.

We say that the function f is ***continuous*** *if f is continuous at each point $\mathbf{x}$ in its domain.*

REMARK 5.17. Recall that for a function $f(x)$ of a single variable is continuous if its graph is an unbroken curve. Similarly, a funciton $f(\mathbf{x})$ of several variables is continuous if it is an unbroken surface (i.e., a surfaces without holes or tears)

Let me conclude this lecture by stating some fundamental properties of continuous functions.

THEOREM 5.18. *Suppose $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ and $g : A \subset \mathbb{R}^n \to \mathbb{R}^m$ are continuous functions. Then so are*

1. $cf(\mathbf{x})$ *, where c is any constant.*
2. $f(\mathbf{x}) + g(\mathbf{x})$
3. $f(\mathbf{x}) \cdot g(\mathbf{x})$

THEOREM 5.19. *Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and $g : \mathbb{R}^m \to \mathbb{R}^p$ is continuous then so is the composed function $g \circ f : \mathbb{R}^n \to \mathbb{R}^p$*

THEOREM 5.20. *Suppose $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is continuous and nowhere zero on A. Then the quotient $1/f(\mathbf{x})$ is continuous.*

THEOREM 5.21. *Suppose $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ and*

$$f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x}))$$

Then f is continuous if and only if each

$$f_i : A \subset \mathbb{R}^n \to \mathbb{R}$$

is continuous.