LECTURE 3

Vectors and Vector Spaces, Cont'd

1. Additional Properties of Dot Products

THEOREM 3.1. *For any* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ *and any* $\alpha, \beta \in \mathbb{R}$

1. $(\alpha\mathbf{u} + \beta\mathbf{v}) \cdot \mathbf{w} = \alpha(\mathbf{u} \cdot \mathbf{w}) + \beta(\mathbf{v} \cdot \mathbf{w})$
2. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
3. $\mathbf{u} \cdot \mathbf{u} \geq 0$
4. $\mathbf{u} \cdot \mathbf{u} = 0 \quad \Rightarrow \quad \mathbf{u} = \mathbf{0}$, *where* $\mathbf{0} \equiv (0, 0, 0)$

THEOREM 3.2. ***(Cauchy-Schartz Inequality)*** *If* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, *then*

$$|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}|\,|\mathbf{v}|$$

proof: Let $a = \mathbf{v} \cdot \mathbf{v}$ and $b = -\mathbf{u} \cdot \mathbf{v}$. If $a = 0$ then $\mathbf{v} \cdot \mathbf{v} = 0$, hence $\mathbf{v} = \mathbf{0}$ by the first theorem above. The inequality is thus trivially satisfied since both sides vanish identically when $\mathbf{v} = (0, 0, 0)$. Now suppose $a \neq 0$. By the preceding theorem we have

$$\begin{aligned}
0 &\leq |a\mathbf{u} + b\mathbf{v}| = (a\mathbf{u} + b\mathbf{v}) \cdot (a\mathbf{u} + b\mathbf{v}) \\
&= a^2 (\mathbf{u} \cdot \mathbf{u}) + 2ab(\mathbf{u} \cdot \mathbf{v}) + b^2(\mathbf{v} \cdot \mathbf{v}) \\
&= (\mathbf{v} \cdot \mathbf{v})^2 (\mathbf{u} \cdot \mathbf{u}) - 2(\mathbf{v} \cdot \mathbf{v})(\mathbf{u} \cdot \mathbf{v})^2 + (\mathbf{u} \cdot \mathbf{v})^2(\mathbf{v} \cdot \mathbf{v}) \\
&= (\mathbf{v} \cdot \mathbf{v})^2 (\mathbf{u} \cdot \mathbf{u}) - (\mathbf{v} \cdot \mathbf{v})(\mathbf{u} \cdot \mathbf{v})^2
\end{aligned}$$

Dividing the extreme sides by $a = (\mathbf{v} \cdot \mathbf{v})$ (which is allowed since we assuming at this point that $a \neq 0$), we obtain

$$0 \leq (\mathbf{v} \cdot \mathbf{v})\,(\mathbf{u} \cdot \mathbf{u}) - (\mathbf{u} \cdot \mathbf{v})^2$$

or

$$(\mathbf{u} \cdot \mathbf{v})^2 \leq (\mathbf{v} \cdot \mathbf{v})\,(\mathbf{u} \cdot \mathbf{u}) = |\mathbf{v}|^2\,|\mathbf{u}|^2$$

Taking the positive square root of both sides now yields the desired inequality.

THEOREM 3.3. ***(Triangle Inequality)*** *If* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, *then*

$$|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$$

proof: By the preceding theorem

$$\mathbf{u} \cdot \mathbf{v} \leq |\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}|\,|\mathbf{v}|$$

Thus,

$$|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + 2\mathbf{u} \cdot \mathbf{v} + |\mathbf{v}|^2 \leq |\mathbf{u}|^2 + 2\,|\mathbf{u}|\,|\mathbf{v}| + |\mathbf{v}|^2 = (|\mathbf{u}| + |\mathbf{v}|)^2$$

Taking the square root of both sides yields

$$|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$$