LECTURE 2

Vectors and Vector Spaces, Cont'd

1. Equations of Lines and Planes

1.1. Equation of a Line in $\mathbb{R}^3$. There are two common geometrical ways of describing a straight line in a 3-dimensional space.

- Given two distinct points $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{R}^3$, there is a unique line passing through both $\mathbf{p}_1$ and $\mathbf{p}_2$.
- Given one point $\mathbf{p} \in \mathbb{R}^3$ and a direction $\mathbf{v}$, there is a unique line passing through $\mathbf{p}_0$ with the direction $\mathbf{v}$.

In this course, we shall think of a lines sets of points of the following form

$$\mathbf{L} = \left\{ \mathbf{p} \in \mathbb{R}^3 \mid \mathbf{p} = \mathbf{p}_0 + \mathbf{v}t \quad , \quad t \in \mathbb{R} \right\} \tag{2.1}$$

The connection with the second geometrical description of a line is evident from the notation. To make the connection with the first geometrical description, all we have to do is set $\mathbf{p}_0 = \mathbf{p}_1$ and set $\mathbf{v} = \mathbf{p}_2 - \mathbf{p}_1$.

If we express the vectors $\mathbf{p}, \mathbf{p}_0$, and $\mathbf{v}$ in terms of components; e.g.

$$\begin{aligned} \mathbf{p} &= (x, y, z) \\ \mathbf{p}_0 &= (x_0, y_0, z_0) \\ \mathbf{v} &= (v_x, v_y, v_z) \end{aligned}$$

the we obtain from (**??**) the following *parametric equation* for a line

$$\begin{aligned} x &= x_0 + v_x t \\ y &= y_0 + v_y t \\ z &= z_0 + v_z t \end{aligned}$$

In terms of the components of two points $\mathbf{p}_1 = (x_1, y_1, z_1)$ and $\mathbf{p}_2 = (x_2, y_2, z_2)$ lying on the line we have corresponding to the first geometrical description of a line the following *parametric equation*

$$\begin{aligned} x &= x_1 + (x_2 - x_1)\,t \\ y &= y_1 + (y_2 - y_1)t \\ z &= z_1 + (z_2 - z_1)t \end{aligned}$$

1.2. Equation of a Plane in $\mathbb{R}^3$. Just as a line can be prescribed by specifying its direction and a single point on the line; a *plane* can be prescribed by specifying a single point $\mathbf{p}_0$ lying in the plane and two distinct directions $\mathbf{v}, \mathbf{u}$ lying in the plane. In vector notation such a prescription takes the form

$$\mathbf{P} = \left\{ \mathbf{p} \in \mathbb{R}^3 \mid \mathbf{p} = \mathbf{p}_0 + \mathbf{u}s + \mathbf{v}t \quad , \quad s, t \in \mathbb{R} \right\}$$

If we set

$$\begin{aligned}\mathbf{p} &= (x, y, z)\\ \mathbf{u} &= (u_x, u_y, u_z)\\ \mathbf{v} &= (v_x, v_y, v_z)\end{aligned}$$

then the relation $\mathbf{p} = \mathbf{p}_0 + \mathbf{u}s + \mathbf{v}t$ leads to the following *parametric representation* of a plane

$$\begin{aligned}x &= x_0 + u_x s + v_x t\\ y &= y_0 + u_y s + v_y t\\ z &= z_0 + u_z s + v_z t\end{aligned}$$

Another way of prescribing a plane is to specify one point $\mathbf{p}_0$ lying in the plane and the direction of a vector $\mathbf{n}$ that is perpendicular to the plane. If another point $\mathbf{p}_1$ is to lie in the plane the plane, the vector from $\mathbf{p}_0$ to $\mathbf{p}_1$ must be perpendicular to $\mathbf{n}$, since $\mathbf{n}$ is perpendicular to every direction in the plane. In terms of vector notation we must have

$$0 = \mathbf{n} \cdot (\mathbf{p}_1 - \mathbf{p}_0)$$

If we set

$$\begin{aligned}\mathbf{n} &= (n_x, n_y, n_z)\\ \mathbf{p}_1 &= (x, y, z)\\ \mathbf{p}_0 &= (x_0, y_0, z_0)\end{aligned}$$

then we have

$$0 = \mathbf{n} \cdot (\mathbf{p}_1 - \mathbf{p}_0) = n_x(x - x_0) + n_y(y - y_0) + n_z(z - z_0)$$

1.3. Applications.

EXAMPLE 2.1. Find the line passing through the point $(3, 1, -2)$ that intersects the line $\mathbf{l}_0$

$$\begin{aligned}x &= -1 + t\\ y &= -2 + t\\ z &= -1 + t\end{aligned}$$

perpendicularly.

- The vector equation for the line $\mathbf{l}_0$ is

  $$\mathbf{l}_0 = (-1, -2, -1) + t(1, 1, 1)$$

  ans so the direction of the line $\mathbf{l}_0$ is $\mathbf{v}_0 = (1, 1, 1)$. If $\mathbf{l}$ is a line through the point $(3, 1, -2)$ then it has an equation of the form

  $$\mathbf{l} = (3, 1, -2) + t\mathbf{v}$$

  Now if $\mathbf{l}$ intersects $\mathbf{l}_0$ perpendicularly the direction $\mathbf{v}$ of $\mathbf{l}$ must be perpendicular to the direction $(1, 1, 1)$ of $\mathbf{l}_0$. Therefore

  $$\begin{aligned}0 &= \mathbf{v} \cdot (1, 1, 1)\\ &= v_x + v_y + v_z\end{aligned}$$

  We know also have a point (x, y, z) common to both lines so

  $$\begin{aligned}-1 + t = x &= 3 + v_x s\\ -2 + t = y &= 1 + v_y s\\ -1 + t = z &= -2 + v_z s\end{aligned}$$

Note that we can set $s = 1$ if we simulaneously rescale the direction vector $\mathbf{v}$. We thus arrive at four equation for four unknowns

$$\begin{aligned} v_x + v_y + v_z &= 0 \\ v_x - t &= -4 \\ v_y - t &= -3 \\ v_z - t &= 1 \end{aligned}$$

If we sum the last three equations we get

$$v_x + v_y + v_z - 3t = -6$$

or, using the first equation,

$$-3t = -6 \quad \Rightarrow \quad t = 2$$

We then find

$$\begin{aligned} v_x &= -4 + t = -2 \\ v_y &= -3 + t = -1 \\ v_z &= 1 + t = 3 \end{aligned}$$

Thus, $\mathbf{v} = (-2, -1, 3)$ and the equation of the line $\mathbf{l}$ is

$$\mathbf{l} = (3, 1, -2) + t(-2, -1, 3)$$

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EXAMPLE 2.2. Find the equation of the plane that contains the point $(2, -1, 3)$ and is perpendicular to the line

$$\mathbf{l} = (1, -1, 3) + t(3, -2, 4)$$

- If $\mathbf{p} = (x, y, z)$ is a point on this plane, then line from the point $\mathbf{p}_0 = (2, -1, 3)$ to $\mathbf{p}$ will also lie in the plane and so must be perpendicular to the direction $\mathbf{v} = (3, -2, 4)$ of $\mathbf{l}$. This leads to the condition

$$\begin{aligned} 0 &= \mathbf{v} \cdot (\mathbf{p} - \mathbf{p}_0) \\ &= (3, -2, 4) \cdot (x - 2, y - 1, z - 3) \\ &= 3x - 6 - 2y + 2 + 4z - 12 \\ &= 3x - 2y + 4z - 16 \end{aligned}$$

The equation of the plane is thus

$$3x - 2y + 4z = 16$$

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EXAMPLE 2.3. Find the equation of the plane containing the lines

$$\begin{aligned} \mathbf{l}_1 &= (0, 1, 1) + t(1, 2, 1) \\ \mathbf{l}_2 &= (0, 1, 0) + t(1, -1, 1) \end{aligned}$$

- The direction of the first line is $\mathbf{v}_1 = (1, 2, 1)$, the direction of the second line is $\mathbf{v}_2 = (1, -1, 1)$, and the direction that is perpendicular to both these lines is

$$\begin{aligned} \mathbf{n} &= \mathbf{v}_1 \times \mathbf{v}_2 \\ &= (1, 2, 1) \times (1, -1, 1) \\ &= ((2)(1) - (1)(-1), (1)(1) - (1)(1), (1)(-1) - (2)(1)) \\ &= (3, 0, 3) \end{aligned}$$

Every other vector in the plane must also be perpendicular to $\mathbf{n}$.

Since the point $(0,1,1)$ lies in the line $\mathbf{l}_1$ which in turn lies in the plane, $\mathbf{p}_0 = (0,1,1)$ is a point lying in the plane. If $\mathbf{p} = (x,y,z)$ is any other point in the plane, then the displacement vector $\mathbf{p} - \mathbf{p}_0 = (x, y-1, z-1)$ must also lie in the plane and must be perpendicular to $\mathbf{n}$. Therefore

$$\begin{aligned} 0 &= \mathbf{n} \cdot (\mathbf{p} - \mathbf{p}_0) \\ &= (3,0,3) \cdot (x, y-1, z-1) \\ &= 3x + 3z - 3 \end{aligned}$$

The equation of the plane is thus

$$3x + 3z = 3$$

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