Math 4013
Homework Problems from Chapter 6

Section 6.1

6.1.1. Let $S^* = (0,1] \times [0,2\pi)$ and defined $T(r,\theta) = (r\cos(\theta), r\sin(\theta))$. Determine the image set S and show that T is one-to-one on S^*.

6.1.2. Let $D^* = [0,1] \times [0,1]$ and define T on D^* by $T(u,v) = \left(-u^2 + 4u, v\right)$. Find D. Is T one-to-one?

6.1.3. Let $D^* = [0,1] \times [0,1]$ and define T on D^* by $T(u,v) = (uv, u)$. Find D. Is T one-to-one? If not, can we eliminate some subset of D^* so that on the remainder T is one-to-one?

6.1.4. Let $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is a 2×2 matrix. Show that T is one-to-one if and only if the determinant of $\mathbf{A}$ is non-zero.

6.1.5. Suppose $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is linear; i.e, $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a 2×2 matrix. Show that if $\mathit{det}\,\mathbf{A} \neq 0$, then T takes parallelograms to parallelograms. (Hint: any parallelogram in $\mathbb{R}^2$ can be described as a set $\{\mathbf{r} = \mathbf{p} + \lambda\mathbf{v} + \mu\mathbf{w} \mid \lambda, \mu \in [0,1]\}$ where $\mathbf{p}$, $\mathbf{v}$, $\mathbf{w}$ are suitable vectors in $\mathbb{R}^2$ with $\mathbf{v}$ not a scalar multiple of $\mathbf{w}$.

Section 6.2

6.2.1. Let D be the unit circle. Evaluate

$$\int_D \exp\left(x^2 + y^2\right) dx\, dy$$

by making a change of variables to polar coordinates.

6.2.2. Let D be the region $0 \le y \le x$ and $0 \le x \le 1$. Evaluate

$$\int_D (x+y) dx\, dy$$

by making the change of variables $x = u + v$, $y = u - v$. Check your answer by evaluating the integral directly by using an iterated integral.

6.2.3. Let $T(u,v) = (x(u,v), y(u,v))$ be the mapping defined by $T(u,v) = (4u, 2u + 3v)$. Let D^* be the region in $u - v$ plane corresponding to the rectangle $[0,1] \times [1,2]$. Find $D = T(D^*)$ and evaluate

(a) $\int_D xy\, dA$

(b) $\int_D (x-y) dA$

6.2.4. Define $T(u,v) = \left(u^2 - v^2, 2uv\right)$. Let D^* be the set of (u,v) with $u^2 + v^2 \le 1$, $u \ge 0$, $v \ge 0$. Find $T(D^*) = D$. Evaluate

$$\int_D dA \quad .$$

6.2.5. Let $T(u,v)$ be as in Exercise 6.2.4. By making this change of variables evaluate

$$\int_D \frac{dA}{\sqrt{x^2 + y^2}}$$

6.2.6. Integrate $ze^{x^2+y^2}$ over the cylinder $x^2 + y^2 \le 4$, $-2 \le z \le 3$.