Math 4013
Homework Problems from Chapter 4

Section 4.2

4.2.1. Calculate the arc length of the following curves.

(a) $\sigma(t) = \left(6t, 3t^2, t^3\right) \quad , \quad t \in [0, 1]$

(b) $\sigma(t) = \left(\sin(3t), \cos(3t), 2t^{3/2}\right) \quad , \quad t \in [0, 1]$

4.2.2. Let σ be the path $\sigma(t) = (t, t\sin(t), t\cos(t))$. Find the arc length of σ between (0,0,0) and $(\pi, 0, -\pi)$.

Section 4.3

4.3.1. A particle of mass m moves along a path $\mathbf{r}(t)$ according to Newton's law in a force field $\mathbf{F} = -\nabla V$ on $\mathbb{R}^3$, where V is a given potential energy function.

(a) Prove that in the energy along the trajectory

$$E = \frac{1}{2}m\|\mathbf{r}'(t)\|^2 + V(\mathbf{r}(t))$$

is constant in time.

(b) If the particle moves on an equipotential surface, show that its speed is constant.

4.3.2. Sketch a few flow lines of the vector field $\mathbf{F}(x, y) = (x, -y)$.

4.3.3. Let $\mathbf{c}(t)$ be a flow line of a gradient field $\mathbf{F} = -\nabla V$. Prove that $V(\mathbf{c}(t))$ is a decreasing function of t. Explain.

4.3.4. Sketch the gradient field $-\nabla V$ for $V(x, y) = (x + y) / \left(x^2 + y^2\right)$. Sketch the equipotential surface $V = 1$.

4.3.5. Show that $\sigma(t) = \left(e^{2t}, \ln|t|, 1/t\right)$ for $t \neq 0$ is a flow line of the velocity vector field $\mathbf{F}(x, y, z) = \left(2x, z, -z^2\right)$.

Section 4.4

4.4.1. Compute the curl, $\nabla \times \mathbf{F}$, of each of the following vector fields.

(a) $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

(b) $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

(c) $\mathbf{F}(x, y, z) = \left(x^2 + y^2 + z^2\right)(3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k})$

4.4.2. Compute the divergence of each of the vector fields in Exercise 1.

4.4.3. Let $\mathbf{F}(x, y, z) = 3x^2y\mathbf{i} + \left(x^3 + y^3\right)\mathbf{j}$.

(a) Verify that $\nabla \times \mathbf{F} = 0$.

(b) Find a function f such that $\mathbf{F} = \nabla f$.

(c) Is it true that for a vector field $\mathbf{F}$ such a function can exist only if $\nabla \times \mathbf{F} = 0$?

4.4.4. Show that $\mathbf{F} = y(\cos(x))\,\mathbf{i} + x(\sin(y))\,\mathbf{j}$ is *not* a gradient field.

Section 4.5

4.5.1. Suppose $\nabla \cdot \mathbf{F} = 0$ and $\nabla \cdot \mathbf{G} = 0$. Which of the following vector fields necessarily have zero divergence?

(a) $\mathbf{F} + \mathbf{G}$

(b) $\mathbf{F} \times \mathbf{G}$

(c) $(\mathbf{F} \cdot \mathbf{G})\,\mathbf{F}$

4.5.2 Prove the following identities.

(a) $\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\,\mathbf{G} + (\mathbf{G} \cdot \nabla)\,\mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F})$

(b) $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$

(c) $\nabla \times (f\mathbf{F}) = f(\nabla \times \mathbf{F}) + \nabla f \times \mathbf{F}$

4.5.3. Let $\mathbf{F} = \left(2xz^2, 1, y^3zx\right)$, $\mathbf{G} = \left(x^2, y^2, z^2\right)$, and $f = x^2y$. Compute the following quantities.

(a) ∇f

(b) $\nabla \times \mathbf{F}$

(c) $(\mathbf{F} \cdot \nabla)\,\mathbf{G}$

(d) $\mathbf{F} \cdot (\nabla f)$

(e) $\mathbf{F} \times \nabla f$

4.5.4. Let $\mathbf{F}$ be a general vector field. Does $\nabla \times \mathbf{F}$ have to be perpendicular to $\mathbf{F}$.