Section 4.2

4.2.1. Calculate the arc length of the following curves.
(a) \( \sigma(t) = (6t, 3t^2, t^3) \), \( t \in [0, 1] \)
(b) \( \sigma(t) = (\sin(3t), \cos(3t), 2^{3/2}) \), \( t \in [0, 1] \)

4.2.2. Let \( \sigma \) be the path \( \sigma(t) = (t, t\sin(t), t\cos(t)) \). Find the arc length of \( \sigma \) between \((0,0,0)\) and \((\pi, 0, -\pi)\).

Section 4.3

4.3.1. A particle of mass \( m \) moves along a path \( r(t) \) according to Newton’s law in a force field \( \mathbf{F} = -\nabla V \) on \( \mathbb{R}^3 \), where \( V \) is a given potential energy function.
(a) Prove that in the energy along the trajectory
\[
E = \frac{1}{2}m\|r'(t)\|^2 + V(r(t))
\]
is constant in time.
(b) If the particle moves on an equipotential surface, show that its speed is constant.

4.3.2. Sketch a few flow lines of the vector field \( \mathbf{F}(x, y) = (x, -y) \).

4.3.3. Let \( c(t) \) be a flow line of a gradient field \( \mathbf{F} = -\nabla V \). Prove that \( V(c(t)) \) is a decreasing function of \( t \). Explain.

4.3.4. Sketch the gradient field \( -\nabla V \) for \( V(x, y) = (x + y) / (x^2 + y^2) \). Sketch the equipotential surface \( V = 1 \).

4.3.5. Show that \( \sigma(t) = (e^t, \ln |t|, 1/t) \) for \( t \neq 0 \) is a flow line of the velocity vector field \( \mathbf{F}(x, y, z) = (2x, z, -z^2) \).

Section 4.4

4.4.1. Compute the curl, \( \nabla \times \mathbf{F} \), of each of the following vector fields.
(a) \( \mathbf{F}(x, y, z) = xi + yj + zk \)
(b) \( \mathbf{F}(x, y, z) = yzi + xzj + xyk \)
(c) \( \mathbf{F}(x, y, z) = \left(x^2 + y^2 + z^2\right) (3i + 4j + 5k) \)

4.4.2. Compute the divergence of each of the vector fields in Exercise 1.

4.4.3. Let \( \mathbf{F}(x, y, z) = 3x^2yi + (x^3 + y^3)j \).
(a) Verify that \( \nabla \times \mathbf{F} = 0 \).
(b) Find a function \( f \) such that \( \mathbf{F} = \nabla f \).
(c) Is it true that for a vector field \( \mathbf{F} \) such a function can exist only if \( \nabla \times \mathbf{F} = 0 \)?
4.4.4. Show that \( F = y \cos(x) \hat{i} + x \sin(y) \hat{j} \) is not a gradient field.

Section 4.5

4.5.1. Suppose \( \nabla \cdot F = 0 \) and \( \nabla \cdot G = 0 \). Which of the following vector fields necessarily have zero divergence?
(a) \( F + G \)
(b) \( F \times G \)
(c) \( (F \cdot G) F \)

4.5.2 Prove the following identities.
(a) \( \nabla (F \cdot G) = (F \cdot \nabla) G + (G \cdot \nabla) F + F \times (\nabla \times G) + G \times (\nabla \times F) \)
(b) \( \nabla \cdot (F \times G) = G \cdot (\nabla \times F) - F \cdot (\nabla \times G) \)
(c) \( \nabla \times (fF) = f(\nabla \times F) + \nabla f \times F \)

4.5.3. Let \( F = (2xz^2, 1, y^3), G = (x^2, y^2, z^2) \), and \( f = x^2y \). Compute the following quantities.
(a) \( \nabla f \)
(b) \( \nabla \times F \)
(c) \( (F \cdot \nabla) G \)
(d) \( F \cdot (\nabla f) \)
(e) \( F \times \nabla f \)

4.5.4. Let \( F \) be a general vector field. Does \( \nabla \times F \) have to be perpendicular to \( F \).