**Math 4013**
**Homework Problems for Chapter 3**

**Section 3.1**

**3.1.1.** Compute the second partial derivatives $\partial^2 f/\partial x^2$, $\partial^2 f/\partial x\partial y$, $\partial^2 f/\partial y\partial x$, $\partial^2 f/\partial y^2$ for each of the following functions. Verify Theorem 15 in each case.

(a) $f(x,y) = 2xy/\left(x^2+y^2\right)^2$, $(x,y) \neq 0$.

(b) $f(x,y,z) = e^z + (1/x) + xe^{-y}$ , $x \neq 0$.

**3.1.2.** Let

$$f(x,y) = \begin{cases} xy\left(x^2-y^2\right)/\left(x^2+y^2\right) & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = 0 \end{cases}$$

(a) If $(x,y) \neq 0$, calculate $\partial f/\partial x$ and $\partial f/\partial y$.

(b) Show that

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = 0 = \left.\frac{\partial f}{\partial y}\right|_{(0,0)}$$

(c) Show that

$$\left.\frac{\partial^2 f}{\partial x\partial y}\right|_{(0,0)} = 1 \quad , \quad \left.\frac{\partial^2 f}{\partial y\partial x}\right|_{(0,0)} = -1 \quad .$$

**3.1.3.** A function $u = f(x,y)$ with continuous second partial derivatives satisfying Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is called a *harmonic function*. Show that $u(x,y) = x^3 - 3xy^2$ is harmonic.

**Section 3.2**

**3.2.1.** Determine the second order Taylor formula for $f(x,y) = (x+y)^2$ about (0,0).

**3.2.2.** Determine the second order Taylor formula for $f(x,y) = 1/\left(x^2+y^2+1\right)$ about (0,0).

**3.2.3.** Determine the second order Taylor formula for $f(x,y) = e^{x+y}$ about (0,0).

**3.2.4.** Determine the second order Taylor formula for $f(x,y) = \sin(xy) + \cos(xy)$ about (0,0).

**Section 3.3**

**3.3.1.** Find the critical points of the given function and then determine whether they are local maxima, local minima, or saddle points.

$$f(x,y) = x^2 - y^2 + xy$$

**3.3.2.** Find the critical points of the given function and then determine whether they are local maxima, local minima, or saddle points.

$$f(x,y) = x^2 + y^2 + 2xy$$

**3.3.3.** Find the critical points of the given function and then determine whether they are local maxima, local minima, or saddle points.

$$f(x,y) = e^{1+x^2-y^2}$$

**3.3.4.** Find the critical points of the given function and then determine whether they are local maxima, local minima, or saddle points.

$$f(x,y) = 3x^2 + 2xy + 2x + y^2 + y + 4$$

**3.3.5.** An examination of the function $f : \mathbb{R}^2 \to \mathbb{R}, (x,y) \mapsto \left(y - 3x^2\right)\left(y - x^2\right)$ will give an idea of the difficulty of finding conditions that guarantee that a critical point is a relative extrenum when Theorem 5 fails. Show that

(a) the origin is a critical point of $f$;

(b) $f$ has a relative minimum at (0,0) on every straight line through (0,0); that is, if $g(t) = (at, bt)$, then $f \circ g : \mathbb{R} \to \mathbb{R}$ has a relative minimum at 0, for every choice of $a$ and $b$;

(c) The origin is not a relative minimum of $f$.

**3.3.6.** Let $f(x,y) = x^2 - 2xy + y^2$. Here $D = 0$. Can you say whether the critical points are local minima, local maxima, or saddle points?

**Section 4.3**

**3.4.1.** Find the extrema of $f(x,y,z) = x - y + z$ subject to the constraint $x^2 + y^2 + z^2 = 2$.

**3.4.2.** Find the extrema of $f(x,y) = x$ subject to the constraint $x^2 + 2y^2 = 3$.

**3.4.3.** Find the extrema of $f(x,y) = 3x + 2y$ subject to the constraint $2x^2 + 3y^2 = 3$.

**Section 3.5**

**3.5.1.*** Let $F(x,y) = 0$ define a curve in the $xy$ plane through the point $(x_o, y_o)$. Assume that $(\partial F/\partial y)(x_o, y_o) \neq 0$. Show that this curve can be locally represented by the graph of a function $y = g(x)$. Show that the line orthogonal to $\nabla F(x_o, y_o)$ agrees with the tangent line to the graph of $y = g(x)$.

4.**3.5.2.*** (a) Check directly (i.e., without using Theorem 10) where we can solve $F(x,y) = y^2 + y + 3x + 1 = 0$ for $y$ in terms of $x$.

(b) Check that your answer in part (a) agrees with the answer you expect from the implicit function theorem. Compute $dy/dx$.

**3.5.3.*** Show that $x^3z^2 - z^3yx = 0$ is solvable for $z$ as a function of $(x,y)$ near (1,1,1), but not near the origin. Compute $\partial z/\partial x$ and $\partial z/\partial y$ at (1,1).