Math 4013
Homework Problems for Chapter 2

Section 2.1

1. Sketch the level curves and graphs of the following functions:
   (a)
$$f : \mathbb{R}^2 \to \mathbb{R} \quad , \quad (x,y) \mapsto x - y + 2$$
   (b)
$$f : \mathbb{R}^2 \to \mathbb{R} \quad , \quad (x,y) \mapsto -xy$$
2. (a) Describe the behavior, as c varies, of the level curve $f(x,y) = c$ for the function
$$f(x,y) = x^2 + y^2 + 1 \quad .$$
3. Sketch or describe the level surfaces and a section of the graph of the following function:
$$f : \mathbb{R}^3 \to \mathbb{R} \; ; \quad (x,y,z) \mapsto -x^2 - y^2 - z^2 \quad .$$
4. * Sketch or describe the surface in $\mathbb{R}^3$ corresponding to the equation
$$4x^2 + y^2 = 16 \quad .$$
5. * Sketch or describe the surface in $\mathbb{R}^3$ corresponding to the equation
$$\frac{x}{4} = \frac{y^2}{4} + \frac{z^2}{9} \quad .$$

Section 2.2

1. * Show that the following subset of $\mathbb{R}^2$ is open.
$$B = \left\{ (x,y) \in \mathbb{R}^2 \mid y > 0 \right\} \quad .$$
2. * Prove that if U and V are neighborhoods of $\mathbf{x} \in \mathbb{R}^n$, then so are $U \cup V$ and $V \cap U$.
3. Compute the following limits.
   (a)
$$\lim_{(x,y)\to(0,1)} x^3 y$$
   (b)
$$\lim_{(x,y)\to(0,1)} e^x y$$
   (c)
$$\lim_{x\to 0} \frac{\sin^2(x)}{x}$$
   (d)*
$$\lim_{x\to 0} \frac{\sin^2(x)}{x^2}$$
4. Compute the following limits if they exist.
   (a)
$$\lim_{(x,y)\to(0,0)} \left(x^2 + y^2 + 3 \right)$$
   (b)
$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2 + 2}$$

(c)

$$\lim_{(x,y)\to(0,0)} \frac{e^{xy}}{x+1}$$

(d)

$$\lim_{(x,y)\to(0,0)} \frac{\cos(x)-1-\frac{x^2}{2}}{x^4+y^4}$$

(e)

$$\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2}$$

5. Show that the map

$$f:\mathbb{R}\to\mathbb{R}, \quad x \mapsto \frac{x^2e^x}{2-\sin(x)}$$

is continuous.

Section 2.3

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for each of the following functions.
   (a) $f(x,y)=xy$.
   (b) $f(x,y)=x\cos(x)\cos(y)$.
2. Evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the function $z=\log\left[\sqrt{1+xy}\right]$ at the points (1,2) and (0,0).
3. Find the partial derivatives $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ when $w=xe^{x^2+y^2}$.
4. Show that the function

$$f(x,y)=\frac{2xy}{(x^2+y^2)^2}$$

   is differentiable at each point in its domain.
5. Find the equation of the plane tangent to the surface $z=x^2+y^3$ at (3,1,10).
6. Compute the matrix of partial derivatives $\mathbf{D}f$ of the following function:

$$f:\mathbb{R}^2 \to \mathbb{R}^3,\ f(x,y)=(xe^y+\cos(y),x,x+e^y) \quad .$$

7. Find the equation of the tangent plane to $z=x^2+2y^3$ at (1,1,3).

Section 2.4

1. Find $\sigma'(t)$ and $\sigma'(0)$ for the following path.

$$\sigma(t)=\left(e^t,\cos(t),\sin(t)\right).$$

2. Determine the velocity and acceleration vectors, and the equation of the tangent line for each of the following curves at the specified value of t.
   (a) $\mathbf{r}(t)=6t\mathbf{i}+3t^2\mathbf{j}+t^3\mathbf{k}\ ,\ t=0$
   (b) $\sigma(t)=\left(\sin(3t),\cos(3t),2t^{3/2}\right)\ ,\ t=1$
   (c) $\sigma(t)=\left(\cos^2(t),3t-t^3,t\right)\ ,\ t=0$
   (d) $\sigma(t)=(0,0,t)\ ,\ t=1$
3. Determine the velocity and acceleration vectors, and the equation of the tangent line for each of the following curves at the specified value of t.
   (a) $\mathbf{r}(t)=\cos(t)\mathbf{i}+\sin(t)\mathbf{j}\ ,\ t=0$
   (b) $\sigma(t)=\left(t\sin(t),t\cos(t),\sqrt{3}t\right)\ ,\ t=0$
   (c) $\mathbf{r}(t)=\sqrt{2}t\mathbf{i}+e^t\mathbf{j}+e^{-t}\mathbf{k}\ ,\ t=0$
   (d) $\sigma(t)=t\mathbf{i}+t\mathbf{j}+\frac{2}{3}t^{3/2}\mathbf{k}\ ,\ t=9$
4. Find the path σ such that $\sigma(0)=(0,-5,1)$ and $\sigma'(t)=\left(t,e^t,t^2\right)$.

5. Suppose a particle follows a path $\sigma(t) = (e^t, e^{-t}, \cos(t))$ until it flies off on a tangent at time $t = 1$. Where is it at time $t = 2$.

Section 2.5

1. Write out the chain rule for each of the following functions and justify your answer in case using Theorem 11.
   (a) $\frac{\partial h}{\partial x}$ where $h(x, y) = f(x, u(x, y))$.
   (b) $\frac{dh}{dx}$ where $h(x) = f(x, u(x), v(x))$.
   (c) $\frac{\partial h}{\partial x}$ where $h(x, y, z) = f(u(x, y, z), v(x, y), w(x))$.
2. Verify the first special case of the chain rule for the composition $f \circ \mathbf{c}$ in each of the following cases.
   (a) $f(x, y) = xy \quad , \quad \mathbf{c}(t) = (e^t, \cos(t)) \quad .$
   (b) $f(x, y) = e^{xy} \quad , \quad \mathbf{c}(t) = \left(3t^2, t^3\right) \quad .$
3. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be differentiable. Making the substitution

$$x = \rho \cos(\theta) \sin(\phi) \quad , \quad y = \rho \sin(\theta) \sin(\phi) \quad , \quad z = \rho \cos(\phi)$$

   (spherical coordinates) into $f(x, y, z)$, compute $\partial f / \partial \rho$, $\partial f / \partial \theta$, and $\partial f / \partial \phi$.
4. Let $f(u, v) = \left(\tan(u - 1) - e^v, u^2 - v^2\right)$ and $g(x, y) = (e^{x-y}, x - y)$. Calculate $f \circ g$ and $D(f \circ g)(1, 1)$.

Section 2.6

1. Show that the directional derivative of $f(x, y, z) = z^2 x + y^3$ at (1,1,2) in the direction $\left(1/\sqrt{5}, 2/\sqrt{5}, 0\right)$ is $2\sqrt{5}$.
2. Find the equation of the plane tangent to the surface $z = f(x, y)$ at the indicated point.
   (a) $z = x^3 + y^3 - 6xy$, (1,2,-3).
   (b) $z = \cos(x) \cos(y)$, $(0, \pi/2, 0)$.
3. Compute the gradient ∇f for each of the following functions.
   (a) $f(x, y, z) = 1/\sqrt{x^2 + y^2 + z^2}$
   (b) $f(x, y, z) = xy + yz + xz$
   (c) $f(x, y, z) = \frac{1}{x^2 + y^2 + z^2}$.
4. For each of the functions in Exercise 6, what is the direction of fastest increase at (1,1,1)?
5. Captain Ralph is in trouble near the sunny side of Mercury. The temperature of the ship's hull when he is at location (x, y, z) will be given by

$$T(x, y, z) = e^{-x^2 - 2y^2 - 3z^2}$$

   where x, y, z are measured in meters. He is currently at (1,1,1).
   (a) In what direction should he proceed in order to decrease his temperature most rapidly?
   (b) If the ship travels at e^8 meters per second, how fast will the temperature be decreasing if he heads in that direction?
   (c) Unfortunately, the metal of the hull will crack if cooled at a rate greater than $\sqrt{14} e^2$ degrees per second. Describe the set of possible directions in which he may proceed to bring the temperature down at no more than that rate.
6. Compute the second partial derivatives $\partial^2 f / \partial x^2$, $\partial^2 f / \partial x \partial y$, $\partial^2 f / \partial y \partial x$, $\partial^2 f / \partial y^2$ for each of the following functions. Verify Theorem 15 in each case.
   (a) $f(x, y) = 2xy / \left(x^2 + y^2\right)^2$, $(x, y) \neq 0$.
   (b) $f(x, y, z) = e^z + (1/x) + xe^{-y}$, $x \neq 0$.
7. Let

$$f(x, y) = \begin{cases} xy\left(x^2 - y^2\right) / \left(x^2 + y^2\right) & , \quad (x, y) \neq (0, 0) \\ 0 & , \quad (x, y) = 0 \end{cases}$$

   (a) If $(x, y) \neq 0$, calculate $\partial f / \partial x$ and $\partial f / \partial y$.

(b) Show that

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = 0 = \left.\frac{\partial f}{\partial y}\right|_{(0,0)}$$

(c) Show that

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(0,0)} = 1 \quad , \quad \left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(0,0)} = -1 \quad .$$

8. A function $u = f(x, y)$ with continuous second partial derivatives satisfying Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is called a *harmonic function*. Show that $u(x, y) = x^3 - 3xy^2$ is harmonic.