Solutions to Homework Set 1

1. Prove that

"not-\(Q\) ⇒ not-\(P\)" implies "\(P\) ⇒ \(Q\)."

• In class we proved that

"\(A\) ⇒ \(B\)" implies "not-\(B\) ⇒ not-\(A\)"

Replacing the statement \(A\) by the statement not-\(Q\) and the statement \(B\) by the statement not-\(P\), we have

"not-\(Q\) ⇒ not-\(P\)" implies "not-(not-\(P\)) ⇒ not-(not-\(Q\))".

But

\[\text{not-(not-}P\text{)} = P \quad \text{and} \quad \text{not-(not-}Q\text{)} = Q\]

so

"not-\(Q\) ⇒ not-\(P\)" implies "\(P\) ⇒ \(Q\)."

2. Prove that if \(m\) and \(n\) are even integers, then \(n + m\) is an even integer.

• If \(m\) and \(n\) are even integers, then, by definition, there exist integers \(s\) and \(t\) such that

\[m = 2s, \quad n = 2t\]

But then

\[m + n = 2s + 2t = 2(s + t)\]

so \(m + n\) is divisible by 2 as well. Hence, \(m + n\) is even.

3. Prove that if \(n\) is an odd integer, then \(n^2\) is an odd integer.

• We can assume that \(n = 2k + 1\) for some \(k \in \mathbb{Z}\). (The actual justification for this assumption must be postponed until after we discuss the Division Algorithm for \(\mathbb{Z}\).) But then

\[n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1\]

has the form of an odd integer - and so must be odd.

4. Prove that if \(n\) is an integer and \(n^2\) is odd, then \(n\) is odd.

• (Proof by Contradiction). Suppose, on the contrary, that \(n\) is an integer, \(n^2\) is odd, and \(n\) is not odd. Then \(n\) is even, so \(n = 2k\) and

\[n^2 = (2k)^2 = 2(2k^2)\]

is even, which violates our hypothesis.

5. Prove, by the contrapositive method, that if \(c\) is an odd integer then the equation \(n^2 + n - c = 0\) has no integer solution.

• The contrapositive of the proposition is: "If the equation \(n^2 + n - c = 0\) has an integer solution then \(c\) is not an odd integer".

But if \(n\) is an integer, then so is \(n^2 + n\). Hence \(c = n^2 + n\) must be an integer. Suppose \(n\) is even, then there exists a \(k \in \mathbb{Z}\) such that \(n = 2k\). Hence

\[c = (2k)^2 + 2k = 2(2k^2 + k)\]
is even. Suppose on the other hand, that \( n \) is odd. Then there exists a \( s \in \mathbb{Z} \) such that \( n = 2s + 1 \). But then

\[
c = (2s + 1)^2 + (2s + 1) = 4s^2 + 2s + 1 + 2s + 1 = 2(2s^2 + 2s + 1)
\]

is again even. Hence, whether \( n \) is even or odd, \( c \) must be even; i.e., \( c \) must be not-odd.

6. Prove, by mathematical induction, that if \( n \geq 5 \) then \( 2^n > n^2 \).

- Clearly,

\[
2^5 = 32 > 25 = 5^2
\]

We thus only need to show that

\[
"2^n > n^2 \text{ and } n \geq 5" \implies 2^{n+1} > (n+1)^2
\]

Assume

\[
2^n > n^2, \quad n \geq 5
\]

Then

\[
2^n + 2^n > n^2 + n^2
\]

or

\[
(0.1) \quad 2^{n+1} = 2(2^n) = (2^n + 2^n) > n^2 + n^2 = 2n^2
\]

It therefore suffices to check that

\[
(0.2) \quad 2n^2 > (n+1)^2
\]

Expanding, the right hand side of (0.1), we get

\[
2n^2 > n^2 + 2n + 1
\]

or equivalently

\[
(0.3) \quad n^2 > 2n + 1
\]

Statement (eq-0.6.3) is certainly true for \( n = 5 \), since \( 25 > 11 \). But note also that if \( n^2 > 2n + 1 \), then adding \( 2n + 1 \) to both sides yields

\[
n^2 + 2n + 1 > 2n + 1 + 2n + 1
\]

or

\[
(n + 1)^2 > 2n + 2(n + 1) > 2(n + 1) + 1 \quad \text{if } n > 1
\]

Hence statement (0.3) is thus proved by mathematical induction.

This then implies the validity of (0.2). Combining (0.1) and (0.2) we have

\[
2^{n+1} > 2n^2 > (n+1)^2
\]

hence the original statement is proved via the principal of mathematical induction.

7. Prove by the contrapositive method that if \( c \) is an odd integer, there the equation \( n^2 + n + c = 0 \) has no integer solution for \( n \).

- The premise of this statement is

\[
c \text{ is odd}
\]

and the conclusion is

\[
n^2 + n + c = 0 \text{ has no integer solution}
\]

So the contrapositive of this statement would be

\[
n^2 + n + c = 0 \text{ has an integer solution } \implies c \text{ is not odd}
\]
The equation \( n^2 + n + c = 0 \) implies
\[
c = -n^2 - n.
\]
If \( n \) is integer, then it is either even or odd. If \( n \) is even,
\[
n = 2k
\]
for some \( k \in \mathbb{Z} \), and so
\[
c = -n^2 - n = -4k^2 - 2k = 2(-2k^2 - k)
\]
is even. If \( n \) is odd then
\[
n = 2s + 1
\]
for some \( s \in \mathbb{Z} \), and so
\[
c = -n^2 - n = -(4s^2 + 4s + 1) - (2s + 1) = 2(2s^2 - 3s - 1)
\]
is even. Thus, in either case, \( c \) is not odd. Since the truth of the contrapositive implies the truth of the original statement, the proposition is proved.

8. Prove by mathematical induction that
\[
\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \forall n \in \mathbb{Z}^+.
\]

- Let \( S(n) \) be the statement
\[
\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}
\]
We need to show two things: (i) The statement \( S(1) \) is true. (ii) The truth of statement \( S(n) \) implies the truth of statement \( S(n+1) \).

The first is easy.
\[
\sum_{i=1}^{1} i^2 = 1 = \frac{1(1+1)(2+1)}{6}.
\]

As for (ii), assume that \( S(n) \) is true. Then
\[
\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2
= \frac{n(n+1)(2n+1)}{6} + (n+1)^2
= \frac{n(n+1)(2n+1)+6(n+1)^2}{6}
= \frac{(n+1)(n(2n+1)+6(n+1))}{6}
= \frac{(n+1)(2n^2 + 7n + 6)}{6}
= \frac{(n+1)(n+2)(2n+3)}{6}
= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6},
\]
which is just the statement \( S(n+1) \).

9. Prove the following identities.

(a) \[ B \cap (C \cup D) = (B \cap C) \cup (B \cap D) \]
10. Describe each set in set-builder notation:

(a) All positive real numbers.

\[ \{ x \mid x \in \mathbb{R}, x > 0 \} \]

(b) All negative irrational numbers.

\[ \{ x \mid x \in \mathbb{R}, x < 0, x \notin \mathbb{Q} \} \]

(c) All points in the coordinate plane with rational first coordinate.

\[ \{(x, y) \mid x \in \mathbb{Q}, y \in \mathbb{R} \} \]

(d) All negative even integers greater than $-50$. 

\[ (b) \quad B \cup (C \cap D) = (B \cup C) \cap (B \cup D) \]

\[
B \cap (C \cup D) = \{ z \in B \text{ and } z \in C \cup D \}
= \{ z \in B \text{ and } (z \in C \text{ or } z \in D) \}
= \{ (z \in B \text{ and } z \in C) \text{ or } (z \in B \text{ and } z \in D) \}
= (B \cap C) \cup (B \cap D)
\]

\[
B \cup (C \cap D) = \{ z \in B \text{ or } z \in C \cap D \}
= \{ z \in B \text{ or } (z \in C \text{ and } z \in D) \}
= \{ (z \in B \text{ or } z \in C) \text{ and } (z \in B \text{ or } z \in D) \}
= (B \cup C) \cap (B \cup D)
\]

\[
(C \setminus A) \cup (C \cap A) = \{ z \in C \text{ and } z \notin A \} \cup \{ z \in C \text{ and } z \in A \}
= \{ (z \in C \text{ and } z \notin A) \text{ or } (z \in C \text{ and } z \in A) \}
= \{ z \in C \}
= C
\]
11. Which of the following sets are nonempty?
   
   (a) \( \{ r \in \mathbb{Q} \mid r^2 = 2 \} \)
   
   • This set is empty because neither of the roots of \( r^2 = 2 \) is rational.

   (b) \( \{ r \in \mathbb{R} \mid r^2 + 5r - 7 = 0 \} \)
   
   • The quadratic formula gives us the roots of \( r^2 + 5r - 7 = 0 \). They are \( r_\pm = \frac{-5 \pm \sqrt{25 + 28}}{2} = \frac{-5 \pm \sqrt{53}}{2} \) and they are both real numbers. Therefore this set is non-empty.

   (c) \( \{ t \in \mathbb{Z} \mid 6t^2 - t - 1 = 0 \} \)
   
   • We have \( 6t^2 - t - 1 = (3t + 1)(2t - 1) \) and so the roots of \( 6t^2 - t - 1 = 0 \) are \( t = -\frac{1}{3}, \frac{1}{2} \); neither of which is an integer. Therefore this set is empty.

12. Is \( B \) a subset of \( C \) when
   
   (a) \( B = \mathbb{Z} \) and \( C = \mathbb{Q} \)?
   
   • Yes, every integer is also a rational number.

   (b) \( B = \mathbb{R} \) and \( C = \mathbb{Q} \)?
   
   • The solutions of \( x^2 + 2x - 5 = 0 \) are \( x_\pm = \frac{-2 \pm \sqrt{4 + 20}}{2} = -1 \pm \frac{\sqrt{24}}{2} = -1 \pm \sqrt{6} \); neither of which is an integer. and so \( B \) is not a subset of \( C \).

   (c) \( B = \{ a, b, 7, 9, 11, -6 \} \) and \( C = \mathbb{Q} \)?
   
   • The letters \( a \) and \( b \) are not rational numbers. Hence there are two elements of \( B \) that do not lie in \( C \). Hence \( B \) is not a subset of \( C \).

13. In each part find \( B - C \), \( B \cap C \), and \( B \cup C \).

   (a) \( B = \mathbb{Z} \) and \( C = \mathbb{Q} \).
   
   • Since every element of \( \mathbb{Z} \) is an element of \( \mathbb{Q} \) we have
     
     \[
     B - C = \{ \} \quad B \cap C = B = \mathbb{Z} \quad B \cup C = C = \mathbb{Q}
     \]

   (b) \( B = \mathbb{R} \) and \( C = \mathbb{Q} \).
• In this case, every element of $C = \mathbb{Q}$ is an element of $B = \mathbb{R}$

$$B - C = \{x \mid x \in \mathbb{R}, x \notin \mathbb{Q}\}$$

$$B \cap C = C = \mathbb{Q}$$

$$B \cup C = B = \mathbb{R}$$

(c) $B = \{a, b, c, 1, 2, 3, 4, 5, 6\}$ and $C = \{a, c, 2, 4, 6, 8\}$.

•

$$B - C = \{b, 1, 3, 5\}$$

$$B \cap C = \{a, c, 2, 4, 6\}$$

$$B \cup C = \{a, b, c, 1, 2, 3, 4, 5, 6, 8\}$$

14. Let $A, B$ be subsets of $U$. Prove De Morgan’s laws:

(a) $U - (A \cap B) = (U - A) \cup (U - B)$

- We need to show that $U - (A \cap B)$ is a subset of $(U - A) \cup (U - B)$ and that $(U - A) \cup (U - B)$ is a subset of $U - (A \cap B)$

$$U - (A \cap B) \subseteq (U - A) \cup (U - B)$$

$$x \in U - (A \cap B)$$

$$\Rightarrow \quad x \in U \quad \text{and} \quad x \notin A \cap B$$

$$\Rightarrow \quad x \in U \quad \text{and} \quad (x \notin A \quad \text{or} \quad x \notin B)$$

$$\Rightarrow \quad (x \in U \quad \text{and} \quad x \notin A) \quad \text{or} \quad (x \in U \quad \text{and} \quad x \notin B)$$

$$\Rightarrow \quad (x \in U - A) \quad \text{or} \quad (x \in U - B)$$

$$\Rightarrow \quad x \in (U - A) \cup (U - B)$$

The key step here was in passing from the third line to the fourth, where we employed the “distributive law of logic”:

$$A \quad \text{and} \quad (B \quad \text{or} \quad C) \quad \Rightarrow \quad (A \quad \text{and} \quad B) \quad \text{or} \quad (A \quad \text{and} \quad C)$$

$$\quad \text{and}$$

$$- \quad (U - A) \cup (U - B) \subseteq U - (A \cap B)$$

- The argument we use here is just the reverse of the sequence of arguments we used above:

$$x \in x \in (U - A) \cup (U - B)$$

$$\Rightarrow \quad (x \in U - A) \quad \text{or} \quad (x \in U - B)$$

$$\Rightarrow \quad (x \in U \quad \text{and} \quad x \notin A) \quad \text{or} \quad (x \in U \quad \text{and} \quad x \notin B)$$

$$\Rightarrow \quad x \in U \quad \text{and} \quad (x \notin A \quad \text{or} \quad x \notin B)$$

$$\Rightarrow \quad x \in U \quad \text{and} \quad x \notin A \cap B$$

$$\Rightarrow \quad x \in U - (A \cap B)$$

(b) $U - (A \cup B) = (U - A) \cap (U - B)$
• We need to show that $U - (A \cup B)$ is a subset of $(U - A) \cap (U - B)$ and $(U - A) \cap (U - B)$ is a subset of $U - (A \cup B)$. This time we’ll argue a bit more efficiently using biconditional statements.

$$x \in U - (A \cup B)$$

$\Leftrightarrow$  \hspace{1em} $x \in U$ and $x \notin (A \cup B)$

$\Leftrightarrow$  \hspace{1em} $x \in U$ and ($x \notin A$ and $x \notin B$)

$\Leftrightarrow$  \hspace{1em} ($x \in U$ and $x \notin A$) and ($x \in U$ and $x \notin B$)

$\Leftrightarrow$  \hspace{1em} ($x \in U - A$) and ($x \in U - B$)

$\Leftrightarrow$  \hspace{1em} $x \in (U - A) \cap (U - B)$

15.

(a) Give an example of a function that is injective but not surjective.

• The natural inclusion map $i : \mathbb{Z} \to \mathbb{R}$ is injective but not surjective.

(b) Give an example of a function that is surjective but not injective.

• Let $A$ denote the set of nonzero real numbers, and let $B$ denote the set of positive real numbers. Then

$$f : A \to B : \hspace{1em} x \mapsto x^2$$

is surjective, but not injective ($f(-x) = f(x)$, but $x \neq -x$).

16. Prove that $f : \mathbb{R} \to \mathbb{R}$ : $f(x) = x^3$ is injective.

• According to Descartes’ Sign Rule, the number of real roots of a polynomial equation is less than or equal to the number of sign changes in the coefficients. So, the number of real solutions of $x^3 = C$ is less than or equal to 1. If $(x_1)^3 = (x_2)^3$ then $x_1 = x_2$. Hence, $f$ is injective.

17. Prove that $f : \mathbb{R} \to \mathbb{R}$ : $f(x) = -3x + 5$ is surjective.

• Let $y$ be an arbitrary element of the range of $f$. We need to show that there is an $x \in \mathbb{R}$ such that $y = f(x)$. We’ll do this constructively by solving the equation $y = -3x + 5$ for $y$. One has

$$y = -3x + 5 \Leftrightarrow \hspace{1em} \frac{1}{3}(y - 5) = x$$

and so for any $y \in \mathbb{R}$

$$y = f \left( \frac{1}{3}(y - 5) \right) \in \text{Im}(f)$$

18.

Let $B$ and $C$ be nonempty sets. Prove that the function

$$f : B \times C \to C \times B$$
given by $f(x, y) = (y, x)$ is a bijection.

• (i) $f$ is a injection.

Suppose $f(x_1, y_1) = f(x_2, y_2)$. Then $(y_1, x_1) = (y_2, x_2)$, so $y_1 = y_2$ and $x_1 = x_2$, hence $(x_1, y_1) = (x_2, y_2)$. 
• (ii) $f$ is a surjection.

Consider an arbitrary element $(y, x) \in C \times B$. Evidently, $(y, x) = f(x, y)$, so $(y, x) \in \text{Image}(f)$. Hence, $f$ is surjective.