Solutions to Homework Set 1

1. Prove that

$$\text{“not-}Q \;\Rightarrow\; \text{not-}P\text{” implies “}P \;\Rightarrow\; Q\text{”}.$$

- In class we proved that

$$\text{“}A \;\Rightarrow\; B\text{” implies “not-}B \;\Rightarrow \text{not-}A\text{”}$$

Replacing the statement A by the statement not-Q and the statement B by the statement not-P, we have

$$\text{“not-}Q \;\Rightarrow \text{not-}P\text{” implies “not-(not-}P) \;\Rightarrow\; \text{not-(not-}Q)\text{”}.$$

But

$$\text{not-(not-}P) = P \qquad \text{and} \qquad \text{not-(not-}Q) = \text{Q} \quad ,$$

so

$$\text{“not-}Q \;\Rightarrow \text{not-}P\text{” implies “}P \;\Rightarrow\; Q\text{”}.$$

∎

2. Prove that if m and n are even integers, then $n + m$ is an even integer.

- If m and n are even integers, then, by definition, there exist integers s and t such that

$$m = 2s \qquad , \qquad n = 2t \qquad .$$

But then

$$m + n = 2s + 2t = 2(s + t) \quad ,$$

so $m + n$ is divisible by 2 as well. Hence, $m + n$ is even. ∎ ∎

3. Prove that if n is an odd integer, then n^2 is an odd integer.

- We can assume that $n = 2k + 1$ for some $k \in \mathbb{Z}$. (The actual justication for this assumption must be postponed until after we discuss the Division Algorithm for $\mathbb{Z}$). But then

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2\left(2k^2 + 2k\right) + 1$$

has the form of an odd integer - and so must be odd. ∎ ∎

4. Prove that if n is an integer and n^2 is odd, then n is odd.

- (Proof by Contradiction). Suppose, on the contrary, that n is an integer, n^2 is odd, **and** n is not odd. Then n is even, so $n = 2k$ and

$$n^2 = (2k)^2 = 2\left(2k^2\right)$$

is even, which violates our hypothesis. ∎

5. Prove, by the contrapositive method, that if c is an odd integer then the equation $n^2 + n - c = 0$ has no integer solution.

- The contrapositive of the proposition is: “If the equation $n^2 + n - c = 0$ has an integer solution then c is not an odd integer”.

  But if n is an integer, then so is $n^2 + n$. Hence $c = n^2 + n$ must be an integer. Suppose n is even, then there exists a $k \in \mathbb{Z}$ such that $n = 2k$. Hence

$$c = (2k)^2 + 2k = 2\left(2k^2 + k\right)$$

is even. Suppose on the other hand, that n is odd. Then there exists a $s \in \mathbb{Z}$ such that $n = 2s+1$. But then

$$c = (2s+1)^2 + (2s+1) = 4s^2 + 2s + 1 + 2s + 1 = 2\left(2s^2 + 2s + 1\right)$$

is again even. Hence, whether n is even or odd, c must be even; i.e., c must be not-odd. ∎

6. Prove, by mathematical induction, that if $n \geq 5$ then $2^n > n^2$.

- Clearly,

$$2^5 = 32 > 25 = 5^2 \quad .$$

We thus only need to show that

$$\text{“}2^n > n^2 \text{ and } n \geq 5\text{”} \quad \Rightarrow \quad 2^{n+1} > (n+1)^2 \quad .$$

Assume

$$2^n > n^2 \qquad , \qquad n \geq 5 \quad .$$

Then

$$2^n + 2^n > n^2 + n^2$$

or

$$2^{n+1} = 2\left(2^n\right) = \left(2^n + 2^n\right) \quad > \quad n^2 + n^2 = 2n^2 \quad . \tag{0.1}$$

It therefore suffices to check that

$$2n^2 >^? (n+1)^2 \quad . \tag{0.2}$$

Expanding, the right hand side of (0.1), we get

$$2n^2 >^? n^2 + 2n + 1$$

or equivalently

$$n^2 >^? 2n + 1 \quad . \tag{0.3}$$

Statement (eq-0.6.3) is certainly true for $n = 5$, since $25 > 11$. But note also that if $n^2 > 2n+1$, then adding $2n+1$ to both sides yields

$$n^2 + 2n + 1 > 2n + 1 + 2n + 1$$

or

$$(n+1)^2 > 2n + 2(n+1) > 2(n+1) + 1 \quad \text{if} n > 1 \quad .$$

Hence statement (0.3) is thus proved by mathematical induction.
This then implies the validity of (0.2). Combining (0.1) and (0.2) we have

$$2^{n+1} > 2n^2 > (n+1)^2 \quad ,$$

hence the original statement is proved via the principal of mathematical induction. ∎

7. Prove by the contrapositive method that if c is an odd integer, there the equation $n^2 + n + c = 0$ has no integer solution for n.

- The premise of this statement is

$$c \text{ is odd}$$

and the conclusion is

$$n^2 + n + c = 0 \text{ has no integer solution}$$

So the contrapositive of this statement would be

$$n^2 + n + c = 0 \quad \text{has an integer solution} \quad \Rightarrow \quad c \text{ is not odd}$$

The equation $n^2+n+c=0$ implies
$$c=-n^2-n \quad .$$
If n is integer, then it is either even or odd. If n is even,
$$n=2k$$
for some $k\in\mathbb{Z}$, and so
$$c=-n^2-n=-4k^2-2k=2(-2k^2-k)$$
is even. If n is odd then
$$n=2s+1$$
for some $s\in\mathbb{Z}$, and so
$$c=-n^2-n=-\left(4s^2+4s+1\right)-(2s+1)=2\left(2s^2-3s-1\right)$$
is even. Thus, in either case, c is not odd. Since the truth of the contrapositive implies the truth of the original statement, the proposition is proved. ∎

8. Prove by mathematical induction that
$$\sum_{i=1}^{n} i^2=\frac{n(n+1)(2n+1)}{6} \qquad , \qquad \forall\, n\in\mathbb{Z}^+ \quad .$$

- Let $S(n)$ be the statement
$$\sum_{i=1}^{n} i^2=\frac{n(n+1)(2n+1)}{g}$$

We need to show two things. ı(i) The statement $S(1)$ is true. ı(ii) The truth of statement $S(n)$ implies the truth of statement $S(n+1)$.

The first is easy.
$$\sum_{i=1}^{1} i^2=1=\frac{6}{6}=\frac{(1)(1+1)(2+1)}{6} \quad .$$
As for (**??**), assume that $S(n)$ is true. Then
$$\begin{aligned}
\sum_{i=1}^{n+1} i^2 &= \sum_{i=1}^{n} i^2+(n+1)^2 \\
&= \frac{n(n+1)(2n+1)}{6}+(n+1)^2 \\
&= \frac{n(n+1)(2n+1)+6(n+1)^2}{6} \\
&= \frac{(n+1)\left(n(2n+1)+6(n+1)\right)}{6} \\
&= \frac{(n+1)(2n^2+7n+6)}{6} \\
&= \frac{(n+1)(n+2)(2n+3)}{6} \\
&= \frac{(n+1)\left((n+1)+1\right)\left(2(n+1)+1\right)}{6} \qquad ,
\end{aligned}$$
which is just the statement $S(n+1)$. ∎

9. Prove the following identities.

(a) $\quad B\cap(C\cup D)=(B\cap C)\cup(B\cap D)$

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$$\begin{aligned} B \cap (C \cup D) &= \{z \in B \text{ and} z \in C \cup D\} \\ &= \{z \in B \text{ and } (z \in C \text{ or} z \in D)\} \\ &= \{(z \in B \text{ and } z \in C) \text{ or } (z \in B \text{ and} z \in D)\} \\ &= (B \cap C) \cup (B \cap D) \end{aligned}$$

∎

(b) $B \cup (C \cap D) = (B \cup C) \cap (B \cup D)$

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$$\begin{aligned} B \cup (C \cap D) &= \{z \in B \text{ or} z \in C \cap D\} \\ &= \{z \in B \text{ or } (z \in C \text{ and} z \in D)\} \\ &= \{(z \in B \text{ or } z \in C) \text{ and } (z \in B \text{ or} z \in D)\} \\ &= (B \cup C) \cap (B \cup D) \end{aligned}$$

∎

(c) $C = (C - A) \cup (C \cap A)$

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$$\begin{aligned} (C - A) \cup (C \cap A) &= \{z \in C \text{ and} z \notin A\} \cup \{z \in C \text{ and} z \in A\} \\ &= \{(z \in C \text{ and} z \notin A) \text{ or } (z \in C \text{ and} z \in A)\} \\ &= \{z \in C\} \\ &= C \end{aligned}$$

∎

10. Describe each set in set-builder notation:

(a) All positive real numbers.

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$$\{x \mid x \in \mathbb{R} \ , \ x > 0\}$$

∎

(b) All negative irrational numbers.

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$$\{x \mid x \in \mathbb{R} \ , \ x < 0 \ , \ x \notin \mathbb{Q}\}$$

∎

(c) All points in the coordinate plane with rational first coordinate.

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$$\{(x, y) \mid x \in \mathbb{Q} \ , \ y \in \mathbb{R}\}$$

∎

(d) All negative even integers greater than -50.

•
$$\{2k \mid k \in \mathbb{Z}\ ,\ -25 < k < 0\}$$
∎

11. Which of the following sets are nonempty?

(a) $\{r \in \mathbb{Q} \mid r^2 = 2\}$

- This set is empty because neither of the roots of $r^2 = 2$ is rational ∎

(b) $\{r \in \mathbb{R} \mid r^2 + 5r - 7 = 0\}$

- The quadratic formula gives us the roots of $r^2 + 5r - 7 = 0$. They are $r_\pm = \frac{-5\pm\sqrt{25-28}}{2} = -\frac{5}{2} \pm \frac{\sqrt{3}}{2}$ and they are both real numbers. Therefore this set is non-empty. ∎

(c) $\{t \in \mathbb{Z} \mid 6t^2 - t - 1 = 0\}$

- We have $6t^2 - t - 1 = (3t+1)(2t-1)$ and so the roots of $6t^2 - t - 1 = 0$ are $t = -\frac{1}{3}, \frac{1}{2}$; neither of which is an integer. Therefore this set is empty. ∎

12. Is B is a subset of C when

(a) $B = \mathbb{Z}$ and $C = \mathbb{Q}$?

- Yes, every integer is also a rational number. ∎

(b) $B =$ all solutions of $x^2 + 2x - 5 = 0$ and $C = \mathbb{Z}$?

- The solutions of $x^2 + 2x - 5 = 0$ are $x_\pm = \frac{-2\pm\sqrt{4+20}}{2} = -1 \pm \frac{\sqrt{24}}{2} = -1 \pm \sqrt{6}$; neither of which is an integer. and so B is not a subset of C. ∎

(c) $B = \{a, b, 7, 9, 11, -6\}$ and $C = \mathbb{Q}$?

- The letters a and b are not rational numbers. Hence there are two elements of B that do not lie in C. Hence B is not a subset of C. ∎

13. In each part find $B - C$, $B \cap C$, and $B \cup C$.

(a) $B = \mathbb{Z}$ and $\mathbb{C} = \mathbb{Q}$.

- Since every element of $\mathbb{Z}$ is an element of $\mathbb{Q}$ we have

$$\begin{aligned} B - C &= \{\} \\ B \cap C &= B = \mathbb{Z} \\ B \cup C &= C = \mathbb{Q} \end{aligned}$$

∎

(b) $B = \mathbb{R}$ and $\mathbb{C} = \mathbb{Q}$.

- In this case, every element of $C = \mathbb{Q}$ is an element of $B = \mathbb{R}$

$$\begin{aligned} B - C &= \{x \mid x \in \mathbb{R}\ ,\ x \notin \mathbb{Q}\} \\ B \cap C &= C = \mathbb{Q} \\ B \cup C &= B = \mathbb{R} \end{aligned}$$

∎

(c) $B = \{a, b, c, 1, 2, 3, 4, 5, 6\}$ and $C = \{a, c, e, 2, 4, 6, 8\}$.

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$$\begin{aligned} B - C &= \{b, 1, 3, 5\} \\ B \cap C &= \{a, c, 2, 4, 6\} \\ B \cup C &= \{a, b, c, e, 1, 2, 3, 4, 5, 6, 8\} \end{aligned}$$

∎

14. Let A, B be subsets of U. Prove De Morgan's laws:

(a) $U - (A \cap B) = (U - A) \cup (U - B)$

- We need to show that $U - (A \cap B)$ is a subset of $(U - A) \cup (U - B)$ and that $(U - A) \cup (U - B)$ is a subset of $U - (A \cap B)$
  - $U - (A \cap B) \subset (U - A) \cup (U - B)$

$$\begin{aligned} x &\in U - (A \cap B) \\ &\Rightarrow x \in U \quad \mathit{and} \quad x \notin A \cap B \\ &\Rightarrow x \in U \quad \mathit{and} \quad (x \notin A \quad \mathit{or} \quad x \notin B) \\ &\Rightarrow (x \in U \quad \mathit{and} \quad x \notin A) \quad \mathit{or} \quad (x \in U \quad \mathit{and} \quad x \notin B) \\ &\Rightarrow (x \in U - A) \quad \mathit{or} \quad (x \in U - B) \\ &\Rightarrow x \in (U - A) \cup (U - B) \end{aligned}$$

The key step here was in passing from the third line to the fourth, where we employed the "distributive law of logic":

$$A \quad \mathit{and} \quad (B \quad \mathit{or} \quad C) \quad \Leftrightarrow \quad (A \quad \mathit{and} \quad B) \quad \mathit{or} \quad (A \quad \mathit{and} \quad C)$$

  - $(U - A) \cup (U - B) \subset U - (A \cap B)$
    The argument we use here is just the reverse of the sequence of arguments we used above:

$$\begin{aligned} x &\in x \in (U - A) \cup (U - B) \\ &\Rightarrow (x \in U - A) \quad \mathit{or} \quad (x \in U - B) \\ &\Rightarrow (x \in U \quad \mathit{and} \quad x \notin A) \quad \mathit{or} \quad (x \in U \quad \mathit{and} \quad x \notin B) \\ &\Rightarrow x \in U \quad \mathit{and} \quad (x \notin A \quad \mathit{or} \quad x \notin B) \\ &\Rightarrow x \in U \quad \mathit{and} \quad x \notin A \cap B \\ &\Rightarrow x \in U - (A \cap B) \end{aligned}$$

(b) $U - (A \cup B) = (U - A) \cap (U - B)$

- We need to show that $U-(A\cup B)$ is a subset of $(U-A)\cap(U-B)$ and $(U-A)\cap(U-B)$ is a subset of $U-(A\cup B)$. This time we'll argue a bit more efficiently using biconditional statements.

$$\begin{array}{ccl} x & \in & U-(A\cup B) \\ & \Leftrightarrow & x\in U \quad and \quad x\notin (A\cup B) \\ & \Leftrightarrow & x\in U \quad and \quad (x\notin A \quad and \quad x\notin B) \\ & \Leftrightarrow & (x\in U \quad and \quad x\notin A) \quad and \quad (x\in U \quad and \quad x\notin B) \\ & \Leftrightarrow & (x\in U-A) \quad and \quad (x\in U-B) \\ & \Leftrightarrow & x\in (U-A)\cap(U-B) \end{array}$$

15.
(a) Give an example of a function that is injective but not surjective.

- The natural inclusion map $i:\mathbb{Z}\ \rightarrow\ \mathbb{R}$ is injective but not surjective. ∎

(b) Give and example of a function that is surjective but not injective.

- Let A denote the set of nonzero real numbers, and let B denote the set of positive real numbers. Then

$$f:A\ \rightarrow\ B \quad ; \quad x\mapsto x^2$$

is surjective, but not injective ($f(-x)=f(x)$, but $x\neq -x$). ∎

16. Prove that $f:\mathbb{R}\rightarrow\mathbb{R} \quad : \quad f(x)=x^3$ is injective.

- According to Descartes' Sign Rule, the number of real roots of a polynomial equation is less than or equal to the number of sign changes in the coefficients. So, the number of real solutions of $x^3=C$ is less than or equal to 1. If $(x_1)^3=(x_2)^3$ then $x_1=x_2$. Hence, f is injective. ∎

17. Prove that $f:\mathbb{R}\rightarrow\mathbb{R} \quad : \quad f(x)=-3x+5$ is surjective.

- Let y be an arbitray element of the range of f. We need to show that there is an $x\in\mathbb{R}$ such that $y=f(x)$. We'll do this constructively by solving the equation $y=-3x+5$ for y. One has

$$y=-3x+5 \quad \Leftrightarrow \quad -\frac{1}{3}(y-5)=x$$

and so for any $y\in\mathbb{R}$

$$y=f\left(-\frac{1}{3}(y-5)\right)\in\mathrm{Im}(f)$$

∎

18.
Let B and C be nonempty sets. Prove that the function

$$f:B\times C\ \rightarrow\ C\times B$$

given by $f(x,y)=(y,x)$ is a bijection.

- (i) f is a injection.
  Suppose $f(x_1,y_1)=f(x_2,y_2)$. Then $(y_1,x_1)=(y_2,x_2)$, so $y_1=y_2$ and $x_1=x_2$, hence $(x_1,y_1)=(x_2,y_2)$.

- (ii) f is a surjection.

  Consider an arbitrary element $(y, x) \in C \times B$. Evidently, $(y, x) = f(x, y)$, so $(y, x) \in Image(f)$. Hence, f is surjective.

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