LECTURE 4

Methods of Proof, Cont'd

1. Review

Last week we discussed a variety of techniques for proving propositions of the form

$$P \Rightarrow Q \quad .$$

Three of these are indicated diagrammatically below:

Direct Proof and the Forward Backward Method

$$P \Rightarrow P_1 \Rightarrow P_2 \Rightarrow \cdots \Rightarrow Q_{-2} \Rightarrow Q_{-1} \Rightarrow Q$$

Proof by Contradiction

$$\left.\begin{array}{r} P \text{ is true} \\ \text{not-}Q \text{ is true} \end{array}\right\} \Rightarrow \cdots \Rightarrow \textit{Contradiction with known results}$$

Contrapositive Method

$$\text{not-}Q \Rightarrow \cdots \Rightarrow \text{not-}P$$

We also discussed **Proof by Construction** as a method useful in propositions involving an existential quantifier (like "there exists at least one"). In this method one explicitly constructs the desired object to confirm its existence.

The next example is basically a proof by contradiction; however it also involves a bit of the construction method.

2. Proof by Mathematical Induction

Let $\mathbb{N}$ denote the set of non-negative integers

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$

with the usual ordering relation

$$a > b \quad \Leftrightarrow \quad a \neq b \text{ and } a - b \in \mathbb{N} \quad .$$

Thus,

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\} \quad .$$

We shall assume this fundamental axiom:

Axiom 4.1. *Well-Ordering Axiom Every non-empty subset of* $\mathbb{N}$ *contains a smallest element.*

This is certainly true when we restrict attention to non-empty subsets containing only a finite number of elements. However, when the subsets considered are infinite there is no means to prove this axiom except by assuming an equivalent axiom. But neither can one prove (without new hypotheses) that this axiom is false; so we will run into no problems down the line by adopting this axiom from the start as a property of the set of natural numbers.

An important consequence of the Well-Ordering Axiom is the method of proof known as mathematical induction. It can be used to prove statements like

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad .$$

Let us denote this statement by $S(n)$. We observe that

$$\begin{array}{lcl} S(0) & \Leftrightarrow & 0 = \frac{0(1)}{2} \\ S(1) & \Leftrightarrow & 1 = \frac{1(2)}{2} \\ S(2) & \Leftrightarrow & 1+2 = \frac{2(3)}{2} \\ S(3) & \Leftrightarrow & 1+2+3 = \frac{3(4)}{2} \end{array}$$

and so the statements $S(n)$ certainly appear to be true for low values of n. The problem is to prove the statement for **all** values of n.

Here is the basic tool for accomplishing this.

THEOREM 4.1. *(The Principle of Mathematical Induction) Suppose that for each nonnegative integer n, a statement $S(n)$ is given. If*

(i) *$S(0)$ is a true statement; and*
(ii) *Whenever $S(k)$ is a true statement, then $S(k+1)$ is also true;*

then $S(n)$ is a true statement for every $n \in \mathbb{N}$.

Proof. Let

$$\mathcal{F} = \{n \in \mathbb{N} \mid S(n) \text{ is } false\}.$$

To prove the theorem, we need to show that the subset $\mathcal{F}$ is empty. We shall use proof by contradiction to do this. Suppose $\mathcal{F}$ is non-empty. Then by the Well-Ordering Axiom S contains a smallest element, say d. Since $S(0)$ is true by hypothesis, and $S(d)$ is false, we must have $d > 0$. Consequently, $d \geq 1$; and so $d - 1 \in \mathbb{N}$. Now

$$d - 1 < d$$

and d is the smallest element of $\mathcal{F}$, so $d - 1 \notin \mathcal{F}$. Therefore,

$$S(d-1) \text{ is true.}$$

However, property (ii), with $k = d - 1$, implies that

$$S(d-1+1) = S(d) \quad \text{is true}$$

This is a contradiction since d is also assumed to be in $\mathcal{F}$. Therefore, $\mathcal{F}$ must be the empty set and the theorem is proved.

We often use this theorem to prove that a whole series of statements $S(n)$, $n = 0, 1, 2, 3$ is true. To do so one simply has to verify the hypotheses (i) and (ii) are satisfied by the set of statements to be proved.

EXAMPLE 4.2. Use mathematical induction to prove

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad \text{for all } n \in \mathbb{N}$$

Well, as we have seen above, the statement for $n = 0$ amounts to

$$0 = \frac{0(2)}{2}$$

which is certainly true. So $S\,(0)$ is true.

Now we must show the *inductive hypothesis*

$$S(k) \text{ is true} \Rightarrow \; S(k+1) \text{ is true} \tag{4.1}$$

is satisfied. So we *presume* that

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2} \tag{4.2}$$

is true and try to show that this implies

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2} \qquad . \tag{4.3}$$

Now the right hand side of (4.3) can be written

$$\begin{aligned} \sum_{i=1}^{k+1} i &= \sum_{i=1}^{k} i + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \qquad\qquad \text{(if (4.2) is true)} \\ &= \frac{k^2+3k+2}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

and so (3) is confirmed. Having satisfied both hypotheses of the Principle of Mathematical Induction, we may conclude that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad , \quad \text{for all } n \in \mathbb{N} \quad .$$