LECTURE 1

# Logic and Proofs

The primary purpose of this course is to introduce you, most of whom are mathematics majors, to the most fundamental skills of a mathematician; the ability to read, write, and understand proofs. This is a course where **proofs** matter more than the material covered.

That said, I should also stress that this is not supposed to be a killer course. Yes, we are going to be rigorous and meticulous; but we will take our time to cover the material. And while we will be often dealing in abstractions; we shall be doing so to develop concrete ways of handling far reaching concepts.

## 1. Basic Logic

**1.1. Statements.** In order to get our bearings, let us begin with a discussion of logic and proof. Much of this discussion will appear as common sense. However, not all common sense is logical, nor does every common sensical argument constitute a proof. For this reason, we must delineate from the start, exactly what constitutes a logical argument.

DEFINITION 1.1. *A* ***statement*** *is a declarative sentence that is either true or false.*

Each of the following sentences is a statement:

Every square has four sides.

$\pi$ is a rational number.

Orange is the best color.

Note that in the last example is a statement, even though one has no means of verifying its truth or falsity (except perhaps by clarifying what one means by "the best").

Let us now lay out the means by which we manipulate statements in a **logical** manner.

**1.2. Compound Statements.** Suppose $P$ and $Q$ are statements. Then we can form new statements by connecting the two statements by the *connectives* "and" and "or".

**If $P$ and $Q$ are statements, then "$P$ and $Q$" is a true statement only if $P$ and $Q$ are both true; otherwise "$P$ and $Q$" is false.**

Thus,

$$\text{“}P \text{ and } Q\text{” is true} \;\Leftarrow\; \left\{ P \text{ is true and } Q \text{ is true.} \right.$$

$$\text{“}P \text{ and } Q\text{” is false} \;\Leftarrow\; \left\{ \begin{array}{l} P \text{ is true and } Q \text{ is false.} \\ Q \text{ is true and } P \text{ is false.} \\ Q \text{ is false and } P \text{ is false.} \end{array} \right.$$

**If $P$ and $Q$ are statements, then "$P$ or $Q$" is a true statement if $P$ is true, $Q$ is true, or "$P$ and $Q$" is true.**

Thus,

$$\text{“}P \text{ or } Q\text{” is true} \Leftarrow \begin{cases} Q \text{ is true and } P \text{ is true.} \\ P \text{ is true and } Q \text{ is false.} \\ Q \text{ is true and } P \text{ is false.} \end{cases}$$

$$\text{“}P \text{ or } Q\text{” is false} \Leftarrow \{Q \text{ is false and } P \text{ is false.}$$

One must be a little careful here, because the logical connective "or" is a particular usage of the English conjuction "or". For in English the conjunction "or" can be used in such a way as to exclude one or the other of two possibilities:

The result of a coin toss is either head *or* tails.

Or it can be used to include two possibilities:

I need 6 or 7 dollars.

In mathematics, one always uses the logical connective "or" in the inclusive sense. So there are always three possibilities if "$P$ or $Q$" is a true mathematical statement:

$$\text{“}P \text{ or } Q\text{” is true} \Leftarrow \begin{cases} P \text{ is true and } Q \text{ is true.} \\ P \text{ is true and } Q \text{ is false.} \\ Q \text{ is true and } P \text{ is false.} \end{cases}$$

Occassionally we'll run into composite statements that have both "or" and "and" conjunctives. In such cases we have to be careful how we associate the conjunctives

$$(S_1 \text{ and } S_2) \text{ or } S_3 \quad \neq \quad S_1 \text{ and } (S_2 \text{ or } S_3)$$

To see what can happen consider the following table

| $S_1$ | $S_2$ | $S_3$ | $(S_1$ and $S_2)$ | $(S_1$ and $S_3)$ | $S_1$ and $(S_2$ or $S_3)$ | $(S_1$ and $S_2)$ or $(S_1$ and $S_3)$ |
|---|---|---|---|---|---|---|
| F | F | F | F | F | F | F |
| F | F | T | F | F | F | F |
| F | T | F | F | F | F | F |
| F | T | T | F | F | F | F |
| T | F | F | F | F | F | F |
| T | F | T | F | T | T | T |
| T | T | F | T | F | T | T |
| T | T | T | T | F | T | T |

Here we have simply listed all the possibilities for the truth or falsity of three statements $S_1, S_2$, and $S_3$ and computed the composite statements "$S_1$ and $(S_2$ or $S_3)$ " and "$(S_1$ and $S_2)$ or $(S_1$ and $S_3)$". Evidently,

$$\text{“}S_1 \text{ and } (S_2 \text{ or } S_3)\text{”} \iff \text{“}(S_1 \text{ and } S_2) \text{ or } (S_1 \text{ and } S_3)\text{”}$$

Similarly, the table

| $S_1$ | $S_2$ | $S_3$ | $(S_2 \text{ and } S_3)$ | $(S_1 \text{ or } S_2)$ | $(S_1 \text{ or } S_3)$ | $S_1 \text{ or } (S_2 \text{ and } S_3)$ | $(S_1 \text{ or } S_2) \text{ and } (S_1 \text{ or } S_3)$ |
|---|---|---|---|---|---|---|---|
| F | F | F | F | F | F | F | F |
| F | F | T | F | F | T | F | F |
| F | T | F | F | T | F | F | F |
| F | T | T | T | T | T | T | T |
| T | F | F | F | T | T | T | T |
| T | F | T | F | T | T | T | T |
| T | T | F | F | T | T | T | T |
| T | T | T | T | T | T | T | T |

shows

$$\text{“}S_1 \text{ or } (S_2 \text{ and } S_3)\text{”} \quad \Longleftrightarrow \quad \text{“}(S_1 \text{ or } S_2) \text{ and } (S_1 \text{ or } S_3)\text{”}$$

In summary,

Lemma 1.2. *Let $S_1$, $S_2$ and $S_3$ be logical statements. Then*

$$\text{“}S_1 \textit{ and } (S_2 \textit{ or } S_3)\text{”} \quad \Longleftrightarrow \quad \text{“}(S_1 \textit{ and } S_2) \textit{ or } (S_1 \textit{ and } S_3)\text{”}$$

*and*

$$\text{“}S_1 \textit{ or } (S_2 \textit{ and } S_3)\text{”} \quad \Longleftrightarrow \quad \text{“}(S_1 \textit{ or } S_2) \textit{ and } (S_1 \textit{ or } S_3)\text{”}$$

Remark 1.3. Note the similarity of these rules with the way multiplication *distributes* over addition:

$$a * (b + c) = (a * b) + (a * c)$$

You can thus think of the first conclusion of Lemma 1.2 as saying the conjuctive “and” distributes over the conjuctive “or” and that the second conclusion is equivalent to saying that the conjuctive “and” distributes over the conjuctive “or”.

### 1.3. Universal Quantifiers.

The following statements contain universal quantifiers.

For all real numbers $x$, $x^2 \neq -1$.
All triangles have three sides.
For each real number $a$, $a^2 \geq 0$.

Notice that in each of the statements above, a property is attributed to **all** members of a set; this is what we mean by a universal quantifier. We'll use the short hand

every $a \in A$ is $B$

to indicate the basic template for a statement with a universal qualifier (even though we have a variety of ways of phrasing such statements).

### 1.4. Existential Quantifiers.

The following statements contain existential quantifiers

Some integers are prime.
There exists a integer between 7.5 and 9.1.
There exists an irrational real number.

Notice that in each of these statements a property is attributed to **at least one** element of a set; this is what one means by a existential quantifier. We'll use the short hand

at least one $a \in A$ is $B$

to indicate the basic template for a statement with an existential qualifier (even though we have a variety of ways of phrasing such statements).

**1.5. Negation.** The **negation**, not-$P$, of a statement $P$ is the statement such that not-$P$ is true exactly when $P$ is false, and not-$P$ is false exactly when $P$ is true.

In most cases you can transform a statement into its negation by inserting a "not" in the appropriate place.

$$\text{A is B.} \longrightarrow \text{A is not B.}$$

The negation of compound statements works as follows:

**The negation of "$P$ and $Q$" is "not-$P$ or not-$Q$".**

**The negation of "$P$ or $Q$" is "not-$P$ and not-$Q$".**

The negation of universal and existential quantifiers works as follows:

**The negation of a statement with a universal quantifier is a statement with an existential quantifier.**

**The negation of a statement with an existential quantifier is a statement with a universal quantifier.**

For example, the negation of the statement

"All crayons are blue″,

which has a universal quantifier is

"Not all crayons are blue"

which if true, would of course imply that at least one crayon was not blue; i.e. a statement with an existential quantifier.

**Note:** Be careful with negatitng statements that *seem* to use an existential qualifier, but without restricting members of a set to satisfy a condition. For example, the statement of "there exist infinitely many primes" is not of the form "There exists at least one $A$ that is $B$ ; and its negation "there does not exists an infinite number of primes" does not involve a universal qualifier.

In summary

$$\begin{array}{rcl}
A \text{ is } B & \xrightarrow{\text{negation}} & A \text{ is not } B \\
A \text{ and } B & \xrightarrow{\text{negation}} & \text{not-}A \text{ or not-}B \\
A \text{ or } B & \xrightarrow{\text{negation}} & \text{not-}A \text{ and not-}B \\
\text{every } a \in A \text{ is } B & \xrightarrow{\text{negation}} & \text{at least one } a \in A \text{ is not-}B \\
\text{at least one } a \in A \text{ is } B & \xrightarrow{\text{negation}} & \text{every } a \in A \text{ is not-}B
\end{array}$$

**1.6. Conditional Statements.** In mathematics one deals primarily with **conditional statements**; that is to say statements of the form

If $P$, then $Q$.

which is written symbolically as

$$P \Rightarrow Q \quad .$$

Such a statement means that the truth of $P$ guarantees the truth of $Q$. More explicitly i $P \Rightarrow Q$ is true if $Q$ is true whenever $P$ is true. i $P \Rightarrow Q$ is false if $Q$ can be false when $P$ is true.

The statement $P$ is called the *hypothesis*, or *premise*, and the statement $Q$ is called the *conclusion*. Here are some examples:

$$\text{If } x \text{ and } y \text{ are integers, then } x + y \text{ is an integer.}$$
$$x \neq 0 \Rightarrow x^2 > 0.$$

There are several ways of phrasing a conditional statement, all of which mean the same thing:

If $P$, then $Q$.

$P$ implies $Q$.

$P$ is sufficient for $Q$.

$Q$ provided that $P$.

$Q$ whenever $P$.

**1.7. The Contrapositive of a Conditional Statement.** The contrapositive of a conditional statement "If $P$, then $Q$" is the conditional statement "If not-$Q$, then not-$P$". For example, the contrapositive of

$$\text{If } x < 6, \text{ then } x < 8$$

is

$$\text{If } x \text{ is not less than 8, then } x \text{ is not less than 6}$$

or, equivalently,

$$\text{If } x \geq 8, \text{ then } x \geq 6.$$

In this example, the truth of the original conditional statement seems to guarantee the truth of its contrapositive. In fact,

**The conditional statement "$P \Rightarrow Q$" is equivalent to its contrapositive "not-$Q \Rightarrow$ not-$P$".**

Let's prove

" $P \Rightarrow Q$" implies "not-$Q \Rightarrow$ not-$P$" .

By hypothesis, if $P$ is true, then $Q$ is true. Suppose not-$Q$ is true. Then $Q$ is false. But then $P$ can not be true, since that would contradict our hypothesis. So not-$P$ must be true.

**1.8. The Converse of a Conditional Statement.** The converse of the conditional statement

$$P \Rightarrow Q$$

is the conditional statement

$$Q \Rightarrow P \quad .$$

It is important to note that **the truth of a conditional statement does not imply the truth of its converse**. For example, it is true that

$$\text{If } x \text{ is an integer, then } x \text{ is a real number;}$$

but the converse of this statement

$$\text{If } x \text{ is a real number, then } x \text{ is an integer,}$$

is certainly not true.

However, there are some situations in which both a conditional statement and its converse are true. For example, both

If the integer $x$ is even, then the integer $x+1$ is odd

and its converse

If integer $x+1$ is odd, then the integer $x$ is even

are true. We can state this fact more succinctly by saying

The integer $x$ is even **if and only if** the integer $x+1$ is odd .

More generally, the statement

$$P \text{ if and only if } Q$$

which may be abbreviated

$$P \text{ iff } Q$$

or

$$P \Leftrightarrow Q$$

means

$$\text{“}P \Rightarrow Q\text{” and “}Q \Rightarrow P\text{”} \quad .$$

"$P$ if and only if $Q$" is called a **biconditional statement.** When "$P \Leftrightarrow Q$" is a true biconditional statement, $P$ is true exactly when $Q$ is true, and so the statements $P$ and $Q$ can be regarded as equivalent statements (when inserted in other statements).

Example 1.4. Negate the following statements.

(a) Every polynomial can be factored.
   - The negation of this statement would be "not every polynomial can be factored". Or "There exists at least one polynomial that cannont be factored".
     But note that one has to be careful about trying to achieve a negation by haphazardly inserting a "not" in the statement. For example, the negation of "Every polynomial can be factored" is not "Every polynomial cannot be factored".
     It might help to remember that the negation of a universal qualifier (like "every") should involve an existential qualifier (like "there exists")

(b) There is at least one solution of $f(x) = 0$.
   - The negation of a statement with an existential qualifier should involve a universal qualifier. However, the most common sensical way of negating "There is at least one solution of $f(x) = 0$" might be "There aren't any solutions of $f(x) = 0$". But this last statement is equivalent to "For all $x$, $f(x) \neq 0$".

(c) $x$ is even and divisible by 3.
   - A negation of statement involving the conjuctive "and" should involve the conjuctive "or". The negation of "$x$ is even and divisible by 3" is "$x$ is not even or not divisible by 3".

(d) Bob lives in Tulsa or Oklahoma City.
   - A negation of statement involving the conjuctive "or" should involve the conjuctive "and". The negation of "Bob lives in Tulsa or Oklahoma City" is "Bob does not live in Tulsa and Bob does not live in Oklahoma City."

Example 1.5. Formulate a contrapositive for each of the following conditional statements.

- If $x$ is divisible by 2 then $x^2$ is divisible by 4
   -
$$\begin{aligned} P &= \text{“}x\text{ is divisible by 2”} \xrightarrow{\text{negation}} not\text{-}P = \text{“}x\text{ is not divisible by 2”} \\ Q &= \text{“}x^2\text{ is divisible by 4”} \xrightarrow{\text{negation}} not\text{-}Q = \text{“}x^2\text{ is not divisible by 4”} \end{aligned}$$

     and so the contrapositive *not*-$Q \Rightarrow$ *not*-$P$ is

     "If $x^2$ is not divisible by 4, then $x$ is not divisible by 2"

- If $x$ is even and divisible by 3 then $x^2$ is divisible by 4 and 9.
  –

$$P = \text{“}x\text{ is even and divisible by 3”} \xrightarrow{\text{negation}} \textit{not-}P = \text{“}x\text{ is not even or not divisible by 2”}$$
$$Q = \text{“}x^2\text{ is divisible by 4 and 9”} \xrightarrow{\text{negation}} \textit{not-}Q = \text{“}x^2\text{ is not divisible by 4 or not divisible by 9”}$$

The contrapositive of the original statement is thus

“If $x^2$ is not divisible by 4 or not divisible by 9, then $x$ is not even or not divisible by 2”

- If there exists one solution of $f' = f$, then there are infinitely many solutions.

$$P = \text{“there is one solution of } f' = f\text{”} \xrightarrow{\text{negation}} \textit{not-}P = \text{“there are no solution of } f' = f\text{”}$$
$$Q = \text{“there are infinitely many solutions”} \xrightarrow{\text{negation}} \textit{not-}Q = \text{“there are not infinitely many solutions”}$$

Thus the contrapositive is

“If there are not infinitely many solutions, then there are no solutions of $f' = f$.

Note that in this example, we are using the existential qualifiers “there is/are” in both $P$ and $Q$, as well as in their negations. Thus, one doesn't always **have to** replace existential qualifiers with universal qualifiers upon negation. In fact, in this example one has to think a bit how to negate $P$ in such a way that its negation uses a universal qualifier. Here's one way:

$$P = \text{“There exists one function satisfying } f' = f\text{”} \xrightarrow{\text{negation}} \textit{not-}P = \text{“Every function satisfies } f' \neq f.\text{”}$$

We can't do a similar thing with the statement $Q$, “there are infinitely many solutions”, because it is not really a statement about the existence of elements with a particular property that have a particular property. Rather it is a statement about the size of a certain set (the set of solutions).

The moral of the last example is that often times is better to simply employ your natural mathematical instincts than try to negate statements by adherring to fixed set of logic rules.