MATH 3613

Homework Problems from Chapter 3

§3.1

3.1.1. The following subsets of $\mathbb{Z}$ (with ordinary addition and multiplication) satisfy all but one of the axioms for a ring. In each case, which axiom fails.
(a) The set S of odd integers.
(b) The set of nonnegative integers.

3.1.2.
(a) Show that the set R of all multiples of 3 is a subring of $\mathbb{Z}$.
(b) Let k be a fixed integer. Show that the set of all multiples of k is a subring of $\mathbb{Z}$.

3.1.3. Let $R = \{0, e, b, c\}$ with addition and multiplication defined by the tables below:

$$\begin{array}{c|cccc} + & 0 & e & b & c \\ \hline 0 & 0 & e & b & c \\ e & e & 0 & c & b \\ b & b & c & 0 & e \\ c & c & b & e & 0 \end{array} \qquad \begin{array}{c|cccc} \cdot & 0 & e & b & c \\ \hline 0 & 0 & 0 & 0 & 0 \\ e & 0 & e & b & c \\ b & 0 & b & e & c \\ c & 0 & c & c & 0 \end{array}$$

Assume distributivity and associativity and show that R is a ring with identity. Is R commutative?

3.1.4. Let $F = \{0, e, a, b\}$ with addition and multiplication defined by the tables below:

$$\begin{array}{c|cccc} + & 0 & e & a & b \\ \hline 0 & 0 & e & a & b \\ e & e & 0 & b & a \\ a & a & b & 0 & e \\ b & b & a & e & 0 \end{array} \qquad \begin{array}{c|cccc} \cdot & 0 & e & a & b \\ \hline 0 & 0 & 0 & 0 & 0 \\ e & 0 & e & a & b \\ a & 0 & a & b & e \\ b & 0 & b & e & a \end{array}$$

Assume distributivity and associativity and show that R is a field.

3.1.5. Which of the following five sets are subrings of $M(\mathbb{R})$. Which ones have an identity?

(a) $$A = \left\{ \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix} \mid r \in \mathbb{Q} \right\}$$

(b) $$B = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$$

(c) $$C = \left\{ \begin{pmatrix} a & a \\ b & b \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$$

(d) $$D = \left\{ \begin{pmatrix} a & 0 \\ a & 0 \end{pmatrix} \mid a \in \mathbb{R} \right\}$$

(e) $$D = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{R} \right\}$$

3.1.6. Let R and S be rings. Show that the subset $\bar{R} = \{(r, 0_S) \mid r \in R\}$ is a subring of $R \times S$.

3.1.7 If R is a ring, show that $R^* = \{(r, r) \mid r \in R\}$ is a subring of $R \times R$.

3.1.8. Is $\{1, -1, i, -i\}$ a subring of $\mathbb{C}$?

3.1.9. Let p be a positive prime and let R be the set of all rational numbers that can be written in the form $\frac{r}{p^i}$ with $r, i \in \mathbb{Z}$. Show that R is a subring of $\mathbb{Q}$.

3.1.10. Let T be the ring of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and let f, g be given by

$$f(x) = \begin{cases} 0 & \text{if } x \le 2 \\ x-2 & \text{if } 2 < x \end{cases} \quad , \quad g(x) = \begin{cases} 2-x & \text{if } x \le 2 \\ 0 & \text{if } 2 < x \end{cases} \quad .$$

Show that $f, g \in T$ and that $fg = 0_T$, and therefore that T is not an integral domain.

3.1.11. Let

$$\mathbb{Q}(\sqrt{2}) = \left\{ r + s\sqrt{2} \mid r, s \in \mathbb{Q} \right\} \quad .$$

Show that $\mathbb{Q}(\sqrt{2})$ is a subfield of $\mathbb{R}$.

3.1.12. Prove Theorem 3.1: If R and S are rings, then the Cartesian product $R \times S$ can be given the structure of a ring by setting

$$\begin{aligned} (r, s) + (r', s') &= (r + r', s + s') \\ (r, s)(r', s') &= (rr', ss') \\ 0_{R \times S} &= (0_R, 0_S) \quad . \end{aligned}$$

Also, if R and S are both commutative, then so is $R \times S$; and if R and S each have an identity, then so does $R \times S$.

3.1.13. Prove or disprove: If R and S are integral domains, then $R \times S$ is an integral domain.

3.1.14. Prove or disprove: If R and S are fields, then $R \times S$ is a field.

§3.2

3.2.1. If R is a ring and $a, b \in R$ then
(a) $(a + b)(a - b) =$?
(b) $(a + b)^3 =$?
(c) What are the answers to (a) and (b) if R is commutative?

3.2.2. An element e of a ring R is said to be **idempotent** if $e^2 = e$.
(a) Find four idempotent elements of the ring $M_2(\mathbb{R})$.
(b) Find all idempotents in $\mathbb{Z}_{12}$.

3.2.3. Prove that the only idempotents in an integral domain R are 0_R and 1_R.

3.2.4. Prove or disprove: The set of units in a ring R with an identity is a subring of R.

3.2.5. (a) If a and b are units in a ring R with identity, prove tha ab is a unit and $(ab)^{-1} = b^{-1}a^{-1}$.
(b)Give an example to show that if a and b are units, then $(ab)^{-1}$ may not be the same as $a^{-1}b^{-1}$. (Hint: consider the matrices $\mathbf{i}$ and $\mathbf{k}$ in the quaterion ring $\mathbb{H}$.)

3.2.6. Prove that a unit in a commutative ring cannot be a zero divisor.

3.2.7.
(a) If ab is a zero divisor in a commutative ring R, prove that a or b is a zero divisor.

(b) If a or b is a zero divisor in a commutative ring R and $ab \neq 0_R$, prove that ab is a zero divisor.

3.2.8. Let S be a non-empty subset of a ring R. Prove that S is a subring if and only if for all $a, b \in S$, both $a - b$ and ab are in S.

3.2.9. Let R be a ring with identity. If there is a smallest integer n such that $n1_R = 0_R$, then n is said to have *characteristic* n. If no such n exists, R is said to have *characteristic zero*. Show that $\mathbb{Z}$ has characteristic zero, and that $\mathbb{Z}_n$ has characteristic n. What is the characteristic of $\mathbb{Z}_4 \times \mathbb{Z}_6$?

§3.3

3.3.1. Let R be a ring and let R^* be the subring of $R \times R$ consisting of all elements of the form (a, a), $a \in R$. Show that the function $f : R \to R^*$ given by $f(a) = (a, a)$ is an isomorphism.

3.3.2. If $f : \mathbb{Z} \to \mathbb{Z}$ is an isomorphism, prove that f is the identity map.

3.3.3. Show that the map $f : \mathbb{Z} \to \mathbb{Z}_n$ given by $f(a) = [a]$ is a surjective homomorphism but not an isomorphism.

3.3.4. If R and S are rings and $f : R \to S$ is a ring homomorphism, prove that

$$f(R) = \{s \in S \mid s = f(a) \text{ for some} a \in R\}$$

is a subring of S

3.3.5.

(a) If $f : R \to S$ and $g : S \to T$ are ring homomorphisms, show that $g \circ f : R \to T$ is a ring homomorphism.

(b) If $f : R \to S$ and $g : S \to T$ are ring isomorphisms, show that $g \circ f : R \to T$ is also a ring isomorphism.

3.3.6. If $f : R \to S$ is an isomorphism of rings, which of the following properties are preserved by this isomorphism? Why?
(a) $a \in R$ is a zero divisor.
(b) R is an integral domain.
(c) R is a subring of $\mathbb{Z}$.
(d) $a \in R$ is a solution of $x^2 = x$.
(e) R is a ring of matrices.

3.3.7. Use the properties that are preserved by ring isomorphism to show that the first ring is not isomorphic to the second.
(a) E (the set of even integers) and $\mathbb{Z}$.
(b) $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and $M_2(\mathbb{R})$.
(c) $\mathbb{Z}_4 \times \mathbb{Z}_{14}$ and $\mathbb{Z}_{16}$.
(d) $\mathbb{Q}$ and $\mathbb{R}$.
(e) $\mathbb{Z} \times \mathbb{Z}_2$ and $\mathbb{Z}$.
(f) $\mathbb{Z}_4 \times \mathbb{Z}_4$ and $\mathbb{Z}_{16}$.