Homework Set 2

(Homework Problems from Chapter 1)

Problems from Section 1.1.

1.1.1. Let n be an integer. Prove that a and c leave the same remainder when divided by n if and only if $a - c = nk$ for some $k \in \mathbb{Z}$.

1.1.2, Let a and c be integers with $c \neq 0$. Then there exist unique integers q and r such that

$$\begin{array}{ll} (i) & a = cq + r \\ (ii) & 0 \leq r < |c| \quad . \end{array}$$

1.1.3. Prove that the square of any integer a is either of the form $3k$ or of the form $3k + 1$ for some integer k.

1.1.4. Prove that the cube of any integer has exactly one of the forms $9k$, $9k + 1$, or $9k + 8$.

Problems from Section 1.2

1.2.1.

(a) Prove that if $a \mid b$ and $a \mid c$ then $a \mid (b + c)$.

(b) Prove that if $a \mid b$ and $a \mid c$, then $a \mid (br + ct)$ for any $r, t \in \mathbb{Z}$.

1.2.2. Prove or disprove that if $a \mid (b + c)$, then $a \mid b$ or $a \mid c$.

1.2.3. Prove that if $r \in \mathbb{Z}$ is a non-zero solution of $x^2 + ax + b = 0$ (where $a, b \in \mathbb{Z}$), then $r \mid b$.

1.2.4. Prove that $GCD(a, a + b) = d$ if and only if $GCD(a, b) = d$.

1.2.5. Prove that if $GCD(a, c) = 1$ and $GCD(b, c) = 1$, then $GCD(ab, c) = 1$.

1.2.6. (a) Prove that if $a, b, u, v \in \mathbb{Z}$ are such that $au + bv = 1$, then $GCD(a, b) = 1$.

(b) Show by example that if $au + bv = d > 0$, then $GCD(a, b)$ need not be d.

Problems from Section 1.3

1.3.1. Let p be an integer other than $0, \pm 1$. Prove that p is prime if and only if for each $a \in \mathbb{Z}$, either $GCD(a, p) = 1$ or $p \mid a$.

1.3.2

Let p be an integer other than 0 ± 1 with this property: Whenever b and c are integers such that $p \mid bc$, then $p \mid c$ or $p \mid b$. Prove that p is prime.

1.3.3. Prove that if every integer integer $n > 1$ can be written in one and only one way in the form

$$n = p_1 p_2 \cdots p_r$$

where the p_i are positive primes such that $p_1 \leq p_2 \leq \cdots \leq p_r$.

1.3.4. Prove that if p is prime and $p \mid a^n$, then $p^n \mid a^n$.

1.3.5.

(a) Prove that there exist no nonzero integers a, b such that $a^2 = 2b^2$.

(b) Prove that $\sqrt{2}$ is irrational.