MATH 3613

Homework Set 1

1. Prove that "not-$Q \Rightarrow$ not-P" implies "$P \Rightarrow Q$"

2. Prove that if m and n are even integers, then $n+m$ is an even integer.

- write $m = 2k$, $n = 2\ell$ and add them.

3. Prove that if n is an odd integer, then n^2 is an odd integer.

- write $n = 2k+1$ and look at the square of the right hand side

4. Prove that if n is an integer and n^2 is odd, then n is odd.

- apply contrapositive method

5. Prove, by the contrapositive method, that if c is an odd integer then the equation $n^2 + n - c = 0$ has no integer solution.

- the equivalent contrapositive statement is: *if n is an integer and $n^2+n = c$ has an integer solution then c is an even integer.* Note that $n^2 + n = n(n+1)$ is always an even integer times an odd integer, hence even.

6. Prove, by mathematical induction, that if $n \geq 5$ then $2^n > n^2$.

- Deduce $2^{n+1} > 2n^2$ from induction hypothesis, and then prove by (another) induction that $2n^2 > (n+1)^2$ for $n \geq 5$.

7. Prove by mathematical induction that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad , \qquad \forall\, n \in \mathbb{Z}^+ \quad .$$

- Write $\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$ and then show that the expression on the far right is equal to $\frac{(n+1)(n+2)(2(n+1)+1)}{6}$

8. Use Truth Tables to prove the following identities.

(a) *S_1 and (S_2 and S_3)* $\iff$ *(S_1 and S_2) and (S_1 and S_3)*

(b) *S_1 or (S_2 and S_3)* $\iff$ *(S_1 or S_2) and (S_1 or S_3)*

(c) *S_1 and (S_2 or S_3)* $\iff$ *(S_1 and S_2) or (S_1 and S_3)*

(d) *S_1 or (S_2 or S_3)* $\iff$ *(S_1 or S_2) or (S_1 or S_3)*

9. Prove the following identities

(a) $\qquad B \cap (C \cup D) = (B \cap C) \cup (B \cap D)$

- We have

$$\begin{aligned} x &\in B\cap C \iff (x\in B)\ \textit{and}\ (x\in C) \\ x &\in B\cup C \iff (x\in B)\ \textit{or}\ (x\in C) \end{aligned}$$

So for an element x to be in $B\cap(C\cup D)$ we must have

$$\begin{aligned} x \in B\cap(C\cup D) &\iff (x\in B)\ \textit{and}\ ((x\in C) \quad \textit{or} \quad (x\in D)) \\ &\iff ((x\in B)\ \textit{and}\ (x\in C)) \quad \textit{or} \quad ((x\in B)\ \textit{and}\ (x\in D)) \quad \text{by 8(c)} \\ &\iff (x\in(B\cap C) \quad \textit{or} \quad (x\in(B\cap D)) \\ &\iff x\in((B\cap C)\cup(B\cap D)) \end{aligned}$$

(b) $\quad B\cup(C\cap D) = (B\cup C)\cap(B\cup D)$

(c) $\quad C = (C-A)\cup(C\cap A)$

9. Describe each set in set-builder notation:

(a) All positive real numbers.

(b) All negative irrational numbers.

(c) All points in the coordinate plane with rational first coordinate.

(d) All negative even integers greater than -50.

10. Which of the following sets are nonempty?

(a) $\left\{r\in\mathbb{Q} \mid r^2=2\right\}$

(b) $\left\{r\in\mathbb{R} \mid r^2+5r-7=0\right\}$

(c) $\left\{t\in\mathbb{Z} \mid 6t^2-t-1=0\right\}$

11. Is B is a subset of C when

(a) $B = \mathbb{Z}$ and $C = \mathbb{Q}$?

(b) $B =$ all solutions of $x^2 + 2x - 5 = 0$ and $C = \mathbb{Z}$?

(c) $B = \{a, b, 7, 9, 11, -6\}$ and $C = \mathbb{Q}$?

12. In each part find $B - C$, $B \cap C$, and $B \cup C$.

(a) $B = \mathbb{Z}$ and $\mathbb{C} = \mathbb{Q}$.

(b) $B = \mathbb{R}$ and $\mathbb{C} = \mathbb{Q}$.

(c) $B = \{a, b, c, 1, 2, 3, 4, 5, 6\}$ and $C = \{a, c, e, 2, 4, 6, 8\}$.

13. Let A, B be subsets of U. Prove De Morgan's laws:

(a) $\qquad U - (A \cap B) = (U - A) \cup (U - B)$

(b) $\qquad U - (A \cup B) = (U - A) \cap (U - B)$

14.
(a) Give an example of a function that is injective but not surjective.

(b) Give and example of a function that is surjective but not injective.

15. Prove that $f : \mathbb{R} \to \mathbb{R} \quad : \quad f(x) = 2x + 1$ is injective.

16. Prove that $f : \mathbb{R} \to \mathbb{R} \quad : \quad f(x) = -3x + 5$ is surjective.

17. Let B and C be nonempty sets. Prove that the function

$$f : B \times C \;\to\; C \times B$$

given by $f(x, y) = (y, x)$ is a bijection.

- Observe any $(c, b) \in C \times B$ is of the form $f((b, c))$ to prove surjectivity.
- Show that $f((b, c)) = f((b', c'))$ requires $b = b'$ and $c = c'$, and hence $(b, c) = (b', c')$ to prove injectivity.