Math 3013
Solutions to Problem Set 9

1. Let $B_0 = \{[1,0,0], [0,1,0], [0,0,1]\}$ be the standard basis for $\mathbb{R}^3$ and let $B_1 = \{[1,1,1], [1,-1,1], [1,1,-1]\}$ be another basis for $\mathbb{R}^3$.

(a) What is the change of basis matrix $\mathbf{C}_{B_1 \to B_0}$ that connects coordinate vectors with respect to B_1 to coordinate vectors with respect to B_0?

- The change of basis matrix that takes coordinate vectors with respect to a basis $B_1 = \{\mathbf{o}_1, \mathbf{o}_2, \mathbf{o}_3\}$ to coordinate vectors with respect to another basis $B_2 = \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ will be
$$\mathbf{C}_{B_1 \to B_2} = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ [\mathbf{o}_1]_{B_2} & [\mathbf{o}_2]_{B_2} & [\mathbf{o}_3]_{B_2} \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$$
where $[\mathbf{o}_i]_{B_2}$ are coordinates of the "old" basis vector $\mathbf{o}_i$ in terms of the new basis B_2.
  In the case at hand, the "old" basis elements $\{[1,1,1], [1,-1,1], [1,1,-1]\}$ are already expressed in terms of the "new" standard basis vectors (each vector of B_1 is written in terms of its standard coordinates; e.g., $[1,1,1] = 1 \cdot \mathbf{e}_1 + 1 \cdot \mathbf{e}_2 + 1 \cdot \mathbf{e}_3$). Thus
$$\mathbf{C}_{B_1 \to B_0} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

(b) Suppose a vector $\mathbf{v}$ has coordinate vector $\mathbf{v}_{B_1} = [1,2,1]$ with respect to the basis B_1. What are its coordinates with respect to the standard basis B_0.

- $$\mathbf{v}_{B_0} = \mathbf{C}_{B_1 \to B_0} \mathbf{v}_{B_1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}$$

(c) What is the change of basis matrix $\mathbf{C}_{B_0 \to B_1}$ that connects standard coordinate vectors (with respect to B_0) to coordinate vectors with respect to B_1.

- We can find $\mathbf{C}_{B_0 \to B_1}$ by computing the inverse of $\mathbf{C}_{B_1 \to B_0}$. One finds
$$\mathbf{C}_{B_0 \to B_1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}$$

(d) If $\mathbf{v}_{B_0} = [2,1,2]$ what are its coordinates with respect to the basis B_1.

- $$\mathbf{v}_{B_1} = \mathbf{C}_{B_1 \to B_0} \mathbf{v}_{B_0} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

2. Suppose $B_1 = \{[1,1], [1,-1]\}$ and $B_2 = \{[2,1], [1,0]\}$. Find the change of basis matrix $\mathbf{C}_{B_1 \to B_2}$ that maps coordinate vectors with respect the basis B_1 to coordinate vectors with respect to the basis B_2.

- We'll do this in two steps. Let $B_0 = \{[1,0], [0,1]\}$ be the standard basis for $\mathbb{R}^2$. The basis elements of B_1 and B_2 are already expressed in terms the standard basis, and so we can immediately write down the following two change of basis matrices

$$\begin{aligned}\mathbf{C}_{B_1\to B_0} &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ \mathbf{C}_{B_2\to B_0} &= \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\end{aligned}$$

We want to go from B_1 to B_2. We can do this using B_0 as an intermediary basis

$$\begin{aligned}\mathbf{C}_{B_1\to B_2} &= \mathbf{C}_{B_0\to B_2}\mathbf{C}_{B_1\to B_0} \\ &= \left(\mathbf{C}_{B_2\to B_0}\right)^{-1}\mathbf{C}_{B_1\to B_0} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}\end{aligned}$$

3. Let $\mathbf{a} = [1,-1,1]$ and $\mathbf{b} = [2,1,1]$.

(a) Find the orthogonal projection of $\mathbf{a}$ onto $\mathbf{b}$.

- $$\mathbf{a}_{||\mathbf{b}} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b} = \frac{(2-1+1)}{4+1+1}[2,1,1] = \left[\frac{2}{3},\frac{1}{3},\frac{1}{3}\right]$$

(b) Find the component of $\mathbf{a}$ that is perpendicular to $\mathbf{b}$.

- $$\mathbf{a}_{\perp\mathbf{b}} = \mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b} = [1,-1,1] - \left[\frac{2}{3},\frac{1}{3},\frac{1}{3}\right] = \begin{bmatrix} \frac{1}{3} & -\frac{4}{3} & \frac{2}{3} \end{bmatrix}$$

4. Let $\mathbf{v}_1 = [1,1,1]$, $\mathbf{v}_2 = [1,1,0]$, $\mathbf{v}_3 = [1,0,0]$. Given that these vectors are linearly independent, apply the Gram-Schmidt orthogonalization process to obtain a corresponding orthogonal basis for $\mathbb{R}^3$.

- $$\begin{aligned}\mathbf{v}_1' &= \mathbf{v}_1 = [1,1,1] \\ \mathbf{v}_2' &= \mathbf{v}_2 - \frac{\mathbf{v}_1'\cdot\mathbf{v}_2}{\mathbf{v}_1'\cdot\mathbf{v}_1'}\mathbf{v}_1' = [1,1,0] - \frac{2}{3}[1,1,1] \\ &= \left[\frac{1}{3},\frac{1}{3},-\frac{2}{3}\right] \\ \mathbf{v}_3' &= \mathbf{v}_3 - \frac{\mathbf{v}_1'\cdot\mathbf{v}_3}{\mathbf{v}_1'\cdot\mathbf{v}_1'}\mathbf{v}_1' - \frac{\mathbf{v}_1'\cdot\mathbf{v}_3}{\mathbf{v}_2'\cdot\mathbf{v}_2'}\mathbf{v}_2' \\ &= [1,0,0] - \left(\frac{1}{3}\right)[1,1,1] - \left(\frac{1}{\left(\frac{6}{9}\right)}\right)\left[\frac{1}{3},\frac{1}{3},-\frac{2}{3}\right] \\ &= \left[\frac{1}{2},-\frac{1}{2},0\right]\end{aligned}$$

5. Let $\mathbf{v}_1 = [1,1,0]$ and $\mathbf{v}_2 = [3,4,2]$

(a) Apply the Gram-Schmidt process to $\{\mathbf{v}_1,\mathbf{v}_2\}$ to an orthogonal basis for the subspace $span(\mathbf{v}_1,\mathbf{v}_2)$ of $\mathbb{R}^3$.

- $$\begin{aligned}\mathbf{v}_1' &= \mathbf{v}_1 = [1,1,0] \\ \mathbf{v}_2' &= \mathbf{v}_2 - \frac{\mathbf{v}_1'\cdot\mathbf{v}_2}{\mathbf{v}_1'\cdot\mathbf{v}_1'}\mathbf{v}_1' = [3,4,2] - \left(\frac{7}{2}\right)[1,1,0] = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 2 \end{bmatrix}\end{aligned}$$

(b) Extend the orthogonal basis found in part (a) to an orthonormal basis for $\mathbb{R}^3$.

- We need to introduce a third linearly independent basis element to get a basis for $\mathbb{R}^3$; $\mathbf{v}_3 = [0,0,1]$ will do (it's easy to see that it lies outside the span of $\mathbf{v}_1'$ and $\mathbf{v}_2'$.)

$$\begin{aligned}\mathbf{v}_3' &= \mathbf{v}_3 - \frac{\mathbf{v}_1'\cdot\mathbf{v}_3}{\mathbf{v}_1'\cdot\mathbf{v}_1'}\mathbf{v}_1' - \frac{\mathbf{v}_1'\cdot\mathbf{v}_3}{\mathbf{v}_2'\cdot\mathbf{v}_2'}\mathbf{v}_2' \\ &= [0,0,1] - \left(\frac{1}{2}\right)[1,1,0] - \left(\frac{2}{\frac{1}{4}+\frac{1}{4}+4}\right)\left[-\frac{1}{2},\frac{1}{2},2\right] \\ &= \begin{bmatrix} \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{bmatrix}\end{aligned}$$

$\{\mathbf{v}_1',\mathbf{v}_2',\mathbf{v}_3'\}$ will be an orthogonal basis. To get an orthonormal basis, we simply have to renormalize the lengths of these vectors:

$$\begin{aligned}\mathbf{u}_1 &= \frac{\mathbf{v}_1'}{\|\mathbf{v}_1'\|} = \left[\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right] \\ \mathbf{u}_2 &= \frac{\mathbf{v}_2'}{\|\mathbf{v}_2'\|} = \left(\sqrt{\frac{2}{9}}\right)\left[-\frac{1}{2},\frac{1}{2},2\right] \\ &= \begin{bmatrix} -\frac{1}{6}\sqrt{2} & \frac{1}{6}\sqrt{2} & \frac{2}{3}\sqrt{2} \end{bmatrix} \\ \mathbf{u}_3 &= \frac{\mathbf{v}_3'}{\|\mathbf{v}_3'\|} = \left(\sqrt{\frac{1}{\frac{9}{81}}}\right)\begin{bmatrix} \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{bmatrix} \\ &= \left[\frac{2}{3},-\frac{2}{3},\frac{1}{3}\right]\end{aligned}$$

(c) Find the orthogonal projection of $[1,2,1]$ onto $W = span(\mathbf{v}_1,\mathbf{v}_2)$

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$$\begin{aligned}\mathbf{v}_\perp &= (\mathbf{v}\cdot\mathbf{u}_1)\,\mathbf{u}_1 + (\mathbf{v}\cdot\mathbf{u}_2)\,\mathbf{u}_2 \\ &= \left[\frac{11}{9},\frac{16}{9},\frac{10}{9}\right]\end{aligned}$$

6. Let $\mathbf{x}_1 = [1,-1,1,1]$, $\mathbf{x}_2 = [1,-1,1,-1]$.

(a) Apply the Gram-Schmidt process to $\{\mathbf{v}_1,\mathbf{v}_2\}$ to an orthogonal basis for the subspace $span(\mathbf{v}_1,\mathbf{v}_2)$ of $\mathbb{R}^4$.

- $$\begin{aligned}\mathbf{v}_1' &= \mathbf{v}_1 = [1,-1,1,1] \\ \mathbf{v}_2' &= \mathbf{v}_2 - \frac{(\mathbf{v}_2\cdot\mathbf{v}_1')}{(\mathbf{v}_1'\cdot\mathbf{v}_1')}\mathbf{v}_1' = \left[\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{3}{2}\right]\end{aligned}$$

(b) Extend the orthogonal basis found in part (a) to an orthonormal basis for $\mathbb{R}^4$.

- We need two more basis vectors to get a basis for $\mathbb{R}^4$. Let's use $\mathbf{v}_3 = [0,0,1,0]$ and $\mathbf{v}_4 = [1,0,0,-1]$. Extending our Gram-Schmidt orthogonal basis, we have

$$\mathbf{v}_3' = \mathbf{v}_3 - \frac{(\mathbf{v}_3\cdot\mathbf{v}_1')}{(\mathbf{v}_1'\cdot\mathbf{v}_1')}\mathbf{v}_1' - \mathbf{v}_2' - \frac{(\mathbf{v}_3\cdot\mathbf{v}_3')}{(\mathbf{v}_2'\cdot\mathbf{v}_2')}\mathbf{v}_2' = \left[-\frac{1}{3},\frac{1}{3},\frac{2}{3},0\right]$$

$$\mathbf{v}_4' = \mathbf{v}_4 - \frac{(\mathbf{v}_4 \cdot \mathbf{v}_1')}{(\mathbf{v}_1' \cdot \mathbf{v}_1')}\mathbf{v}_1' - \mathbf{v}_2' - \frac{(\mathbf{v}_4 \cdot \mathbf{v}_3')}{(\mathbf{v}_2' \cdot \mathbf{v}_2')}\mathbf{v}_2' - \frac{(\mathbf{v}_4 \cdot \mathbf{v}_3')}{(\mathbf{v}_3' \cdot \mathbf{v}_3')} = \left[\frac{1}{2}, \frac{1}{2}, 0, 0\right]$$

The vectors $\mathbf{v}_1', \mathbf{v}_2', \mathbf{v}_3', \mathbf{v}_4'$ now comprise an orthogonal basis. To get an orthonormal basis we must renormalized them so that their lengths are 1. Doing so we get,

$$\begin{aligned}
\mathbf{u}_1 &= \frac{\mathbf{v}_1'}{\|\mathbf{v}_1'\|} = \left[\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] \\
\mathbf{u}_2 &= \frac{\mathbf{v}_2'}{\|\mathbf{v}_2'\|} = \left[\frac{\sqrt{3}}{6}, -\frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{6}, -\frac{\sqrt{3}}{2}\right] \\
\mathbf{u}_3 &= \frac{\mathbf{v}_3'}{\|\mathbf{v}_3'\|} = \left[-\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, 0\right] \\
\mathbf{u}_4 &= \frac{\mathbf{v}_4'}{\|\mathbf{v}_4'\|} = \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0, 0\right]
\end{aligned}$$

(c) Find the orthogonal projection of $[2, 1, 1, 1]$ onto $span\,(\mathbf{v}_1, \mathbf{v}_2)$

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$$\mathbf{v}_{||} = (\mathbf{v} \cdot \mathbf{u}_1)\,\mathbf{u}_1 + (\mathbf{v} \cdot \mathbf{u}_2)\,\mathbf{u}_2 = \left[\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}, 1\right]$$

7. (a) Find an orthogonal basis for $\mathbb{R}^3$ that contains the vector $\mathbf{v} = [3, 1, 5]$.

- We'll just start the Gram-Schmidt process with the vector $\mathbf{v}_1 = [3, 1, 5]$ and two other linearly independent vectors; e.g. $\mathbf{v}_2 = [0, 1, 0]$ and $\mathbf{v}_3 = [0, 0, 1]$. The Gram-Schmidt process applied to $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ yields

$$\begin{aligned}
\mathbf{v}_1' &= [3, 1, 5] \\
\mathbf{v}_2' &= \left[-\frac{3}{35}, \frac{34}{35}, -\frac{1}{7}\right] \\
\mathbf{v}_3' &= \left[-\frac{15}{34}, 0, \frac{9}{34}\right]
\end{aligned}$$

(b) Find a basis for the orthogonal complement of $\mathbf{v}$, in $\mathbb{R}^3$.

- This will just be the span of $\mathbf{v}_2'$ and $\mathbf{v}_3'$

$$\mathbf{v}_1^{\perp} = span\left([3, 1, 5], \left[-\frac{3}{35}, \frac{34}{35}, -\frac{1}{7}\right]\right)$$