Math 3013
Solutions to Problem Set 8

1. Find the characteristic polynomial, the real eigenvalues, and the corresponding eigenvectors for the following matrices.

(a) $\mathbf{A} = \begin{bmatrix} 7 & 5 \\ -10 & -8 \end{bmatrix}$

- The characteristic polynomial is

$$P(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 7-\lambda & 5 \\ -10 & -8-\lambda \end{vmatrix} = (7-\lambda)(-8-\lambda) - (5)(-10) = \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2)$$

The eigenvalues of $\mathbf{A}$ correspond to the roots of $P(\lambda) = 0$; so we have two eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 2$.

The eigenspace corresponding to the eigenvalue $\lambda_1 = -3$ is the solution set of $(\mathbf{A} - (-3)\mathbf{I})\,\mathbf{x} = 0$; i.e., the null space of the matrix

$$\mathbf{A} - (-3)\mathbf{I} = \begin{bmatrix} 7+3 & 5 \\ -10 & -8+3 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ -10 & -5 \end{bmatrix} \xrightarrow[R_2 \to R_2 - R_1]{R_1 \to \frac{1}{5}R_1} \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

The null space of the last matrix is the solution set of

$$\begin{matrix} 2x_1 + x_2 = 0 \\ 0 = 0 \end{matrix} \quad \Rightarrow \quad x_1 = -\frac{1}{2}x_2 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} -\frac{1}{2}x_2 \\ x_2 \end{bmatrix} \in span\left(\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right)$$

Thus, the eigenvectors corresponding to the eigenvalue $\lambda_1 = -3$ are thus of the form

$$\mathbf{v}_1 = r\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \quad , \quad r \in \mathbb{R} - \{0\}$$

The eigenspace corresponding to the eigenvalue $\lambda_2 = 2$ is the solution set of $(\mathbf{A} - (2)\mathbf{I})\,\mathbf{x} = 0$; i.e., the null space of the matrix

$$\mathbf{A} - (-3)\mathbf{I} = \begin{bmatrix} 7-2 & 5 \\ -10 & -8-2 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ -10 & -10 \end{bmatrix} \xrightarrow[R_2 \to R_2 + 2R_1]{R_1 \to \frac{1}{5}R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

The null space of the last matrix is the solution set of

$$\begin{matrix} x_1 + x_2 = 0 \\ 0 = 0 \end{matrix} \quad \Rightarrow \quad x_1 = -x_2 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} \in span\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$$

Thus, the eigenvectors corresponding to the eigenvalue $\lambda_2 = 2$ are thus of the form

$$\mathbf{v}_2 = r\begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad , \quad r \in \mathbb{R} - \{0\}$$

(b) $\mathbf{A} = \begin{bmatrix} -7 & -5 \\ 16 & 17 \end{bmatrix}$

- The characteristic polynomial is

$$P(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -7-\lambda & -5 \\ 16 & 17-\lambda \end{vmatrix} = \lambda^2 - 10\lambda - 39 = (\lambda - 13)(\lambda + 3)$$

The eigenvalues of $\mathbf{A}$ correspond to the roots of $P(\lambda) = 0$; so we have two eigenvalues $\lambda_1 = 13$ and $\lambda_2 = -3$.

The eigenspace corresponding to the eigenvalue $\lambda_1 = 13$ is the solution set of $(\mathbf{A} - (13)\mathbf{I})\,\mathbf{x} = 0$; i.e., the null space of the matrix

$$\mathbf{A} - (13)\mathbf{I} = \begin{bmatrix} -20 & -5 \\ 16 & 4 \end{bmatrix} \xrightarrow[R_2 \to R_2 + \frac{4}{5}R_1]{R_1 \to -\frac{1}{5}R_1} \begin{bmatrix} 4 & 1 \\ 0 & 0 \end{bmatrix}$$

The null space of the last matrix is the solution set of

$$\begin{array}{c} 4x_1 + x_2 = 0 \\ 0 = 0 \end{array} \quad \Rightarrow \quad x_1 = -\frac{1}{4}x_2 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} -\frac{1}{4}x_2 \\ x_2 \end{bmatrix} \in span\left(\begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}\right)$$

Thus, the eigenvectors corresponding to the eigenvalue $\lambda_1 = 13$ are thus of the form

$$\mathbf{v}_1 = r\begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix} \quad , \quad r \in \mathbb{R} - \{0\}$$

The eigenspace corresponding to the eigenvalue $\lambda_2 = -3$ is the solution set of $(\mathbf{A} - (-3)\mathbf{I})\,\mathbf{x} = 0$; i.e., the null space of the matrix

$$\mathbf{A} - (-3)\mathbf{I} = \begin{bmatrix} -4 & -5 \\ 16 & 20 \end{bmatrix} \xrightarrow[R_2 \to R_2 + 4R_1]{R_1 \to -R_1} \begin{bmatrix} 4 & 5 \\ 0 & 0 \end{bmatrix}$$

The null space of the last matrix is the solution set of

$$\begin{array}{c} 4x_1 + 5x_2 = 0 \\ 0 = 0 \end{array} \quad \Rightarrow \quad x_1 = -\frac{5}{4}x_2 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} -\frac{5}{4}x_2 \\ x_2 \end{bmatrix} \in span\left(\begin{bmatrix} -\frac{5}{4} \\ 1 \end{bmatrix}\right)$$

Thus, the eigenvectors corresponding to the eigenvalue $\lambda_2 = -3$ are thus of the form

$$\mathbf{v}_2 = r\begin{bmatrix} -\frac{5}{4} \\ 1 \end{bmatrix} \quad , \quad r \in \mathbb{R} - \{0\}$$

(c) $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}$

- The characteristic polynomial is

$$P(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & -2 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 4$$

The roots of this (quadratic) polynomial are given by the quadratic formula:

$$\lambda = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(4)}}{2(1)} = \frac{3 \pm \sqrt{9-16}}{2} = \frac{3}{2} \pm \frac{\sqrt{-7}}{2} = \frac{3}{2} \pm \sqrt{7}i$$

Thus, we have two complex roots. Lacking a real eigenvalue, the problem ends here.

(d) $\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ -4 & 2 & -1 \\ 4 & 0 & 3 \end{bmatrix}$

- The characteristic polynomial is

$$\begin{aligned} P(\lambda) &= \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -1-\lambda & 0 & 0 \\ -4 & 2-\lambda & -1 \\ 4 & 0 & 3-\lambda \end{vmatrix} \\ &= (-1-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ 0 & 3-\lambda \end{vmatrix} - (0)\begin{vmatrix} -4 & -1 \\ 4 & 3-\lambda \end{vmatrix} + (0)\begin{vmatrix} -4 & 2-\lambda \\ 4 & 0 \end{vmatrix} \\ &= (-1-\lambda)\left((2-\lambda)(3-\lambda) - 0\right) - 0 + 0 \\ &= -(\lambda+1)(\lambda-2)(\lambda-3) \end{aligned}$$

We thus have three eigenvalues: $\lambda_1 = -1$, $\lambda_2 = 2$, and $\lambda_3 = 3$.

The eigenspace corresponding to the eigenvalue $\lambda_1 = -1$ correspond to the null space of

$$\mathbf{A} - (-1)\mathbf{I} = \begin{bmatrix} 0 & 0 & 0 \\ -4 & 3 & -1 \\ 4 & 0 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

i.e. the solution set of

$$\begin{array}{c} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \\ 0 = 0 \end{array} \quad \Rightarrow \quad \begin{array}{c} x_1 = -x_3 \\ x_2 = -x_3 \end{array} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} \in span\left(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right)$$

Thus, the eigenvectors corresponding to the eigenvalue $\lambda_1 = -1$ will be vectors of the form

$$\mathbf{v}_1 = r \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \quad , \quad r \in \mathbb{R} - \{0\}$$

The eigenspace corresponding to the eigenvalue $\lambda_2 = 2$ correspond to the null space of

$$\mathbf{A} - (2)\mathbf{I} = \begin{bmatrix} -3 & 0 & 0 \\ -4 & 0 & -1 \\ 4 & 0 & 1 \end{bmatrix} \quad \rightsquigarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

i.e. the solution set of

$$\begin{array}{c} x_1 = 0 \\ x_3 = 0 \\ 0 = 0 \end{array} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} 0 \\ -x_2 \\ 0 \end{bmatrix} \in span\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

Thus, the eigenvectors corresponding to the eigenvalue $\lambda_2 = 2$ will be vectors of the form

$$\mathbf{v}_2 = r \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad , \quad r \in \mathbb{R} - \{0\}$$

The eigenspace corresponding to the eigenvalue $\lambda_3 = 3$ correspond to the null space of

$$\mathbf{A} - (3)\mathbf{I} = \begin{bmatrix} -4 & 0 & 0 \\ -4 & -1 & -1 \\ 4 & 0 & 0 \end{bmatrix} \quad \rightsquigarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

i.e. the solution set of

$$\begin{array}{c} x_1 = 0 \\ x_2 + x_3 = 0 \\ 0 = 0 \end{array} \quad \Rightarrow \quad \begin{array}{c} x_1 = 0 \\ x_2 = -x_3 \end{array} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} 0 \\ -x_3 \\ x_3 \end{bmatrix} \in span\left(\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right)$$

Thus, the eigenvectors corresponding to the eigenvalue $\lambda_3 = 3$ will be vectors of the form

$$\mathbf{v}_3 = r \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad , \quad r \in \mathbb{R} - \{0\}$$

(e) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ -8 & 4 & -5 \\ 8 & 0 & 9 \end{bmatrix}$

- The characteristic polynomial is

$$P(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ -8 & 4-\lambda & -5 \\ 8 & 0 & 9-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda)(9-\lambda)$$

So we have three possible eigenvalues : $\lambda_1 = 1$, $\lambda_2 = 4$, and $\lambda_3 = 9$. The corresponding eigenvectors are calculated as in the preceding problems:

$$\begin{array}{rclcl} \lambda_1 & = & 1 & \Rightarrow & \mathbf{v}_1 = r\,[-1,-1,1] \quad , \quad r \in \mathbb{R} - \{0\} \\ \lambda_2 & = & 4 & \Rightarrow & \mathbf{v}_2 = r\,[0,1,0] \quad , \quad r \in \mathbb{R} - \{0\} \\ \lambda_3 & = & 9 & \Rightarrow & \mathbf{v}_3 = r\,[0,-1,1] \quad , \quad r \in \mathbb{R} - \{0\} \end{array}$$

(f) $\mathbf{A} = \begin{bmatrix} -4 & 0 & 0 \\ -7 & 2 & -1 \\ 7 & 0 & 3 \end{bmatrix}$

- The characteristic polynomial is

$$P(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -4-\lambda & 0 & 0 \\ -7 & 2-\lambda & -1 \\ 7 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda)(-4-\lambda)$$

So we have three real eigenvalues: $\lambda_1 = 2$, $\lambda_2 = 3$, and $\lambda_3 = -4$. The corresponding eigenvectors (calculated as in the preceding problems) are

$$\begin{aligned} \lambda_1 &= 2 \quad \Rightarrow \quad \mathbf{v}_1 = r\,[0,1,0] \quad , \quad r \in \mathbb{R} - \{0\} \\ \lambda_2 &= 3 \quad \Rightarrow \quad \mathbf{v}_2 = r\,[0,1,-1] \quad , \quad r \in \mathbb{R} - \{0\} \\ \lambda_3 &= -4 \quad \Rightarrow \quad \mathbf{v}_3 = r\,[-1,-1,1] \quad , \quad r \in \mathbb{R} - \{0\} \end{aligned}$$

2. Find the eigenvalues λ_i and the corresponding eigenvectors $\mathbf{v}_i$ for the following linear transformations.

(a) $T([x,y]) = [2x-3y, -3x+2y]$

- First we calculate the matrix corresponding to T:

$$\begin{array}{c} T([1,0]) = [2,-3] \\ T([0,1]) = [-3,2] \end{array} \quad \Rightarrow \quad \mathbf{A}_T = \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$$

The characteristic polynomial for this matrix is

$$P(\lambda) = \det(\mathbf{A}_T - \lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & -3 \\ -3 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda - 5 = (\lambda-5)(\lambda+1)$$

We thus have two real eigenvalues: $\lambda_1 = 5$ and $\lambda_2 = -1$.

The eigenspace corresponding to the eigenvalue $\lambda_1 = 5$ is the null space of

$$\begin{vmatrix} 2-5 & -3 \\ -3 & 2-5 \end{vmatrix} = \begin{vmatrix} -3 & -3 \\ -3 & -3 \end{vmatrix} \quad \rightsquigarrow \quad \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}$$

which is the solution set of

$$\begin{array}{c} x_1 + x_2 = 0 \\ 0 = 0 \end{array} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} \in span\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$$

Hence, the corresponding eigenvectors are of the form

$$\mathbf{v}_1 = r \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad , \quad r \in \mathbb{R} - \{0\}$$

The eigenspace corresponding to the eigenvalue $\lambda_1 = -1$ is the null space of

$$\begin{vmatrix} 2-(-1) & -3 \\ -3 & 2-(-1) \end{vmatrix} = \begin{vmatrix} 3 & -3 \\ -3 & 3 \end{vmatrix} \quad \rightsquigarrow \quad \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix}$$

which is the solution set of

$$\begin{array}{c} x_1 - x_2 = 0 \\ 0 = 0 \end{array} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} \in span\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

Hence, the corresponding eigenvectors are of the form

$$\mathbf{v}_2 = r \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad , \quad r \in \mathbb{R} - \{0\}$$

(b) $T([x_1, x_2, x_3]) = [x_1 + x_3, x_2, x_1 + x_3]$

- First we calculate the matrix corresponding to this linear transfomation:

$$\begin{array}{c} T([1,0,0]) = [1,0,1] \\ T([0,1,0]) = [0,1,0] \\ T([0,0,1]) = [1,0,1] \end{array} \quad \Rightarrow \quad \mathbf{A}_T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

The characteristic polynomial for this matrix is

$$P(\lambda) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = -\lambda(-1+\lambda)(\lambda-2)$$

We thus have three possible eigenvalues: $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_2 = 2$.

The eigenspace corresponding to $\lambda_1 = 0$ will be the null space of

$$\begin{bmatrix} 1-0 & 0 & 1 \\ 0 & 1-0 & 0 \\ 1 & 0 & 1-0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \rightsquigarrow \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

or, equivalently, the solution set of

$$\begin{array}{c} x_1 + x_3 = 0 \\ x_2 = 0 \\ 0 = 0 \end{array} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix}$$

so the correpsponding eigenvectors will be of the form

$$\mathbf{v}_1 = r \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad , \quad r \in \mathbb{R} - \{0\}$$

The eigenspace corresponding to $\lambda_1 = 1$ will be the null space of

$$\begin{bmatrix} 1-1 & 0 & 1 \\ 0 & 1-1 & 0 \\ 1 & 0 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \rightsquigarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

or, equivalently, the solution set of

$$\begin{array}{c} x_1 = 0 \\ x_3 = 0 \\ 0 = 0 \end{array} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix}$$

so the correpsponding eigenvectors will be of the form

$$\mathbf{v}_2 = r \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad , \quad r \in \mathbb{R} - \{0\}$$

The eigenspace corresponding to $\lambda_1 = 2$ will be the null space of

$$\begin{bmatrix} 1-2 & 0 & 1 \\ 0 & 1-2 & 0 \\ 1 & 0 & 1-2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \quad \rightsquigarrow \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

or, equivalently, the solution set of

$$\begin{array}{c} x_1 - x_3 = 0 \\ x_2 = 0 \\ 0 = 0 \end{array} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} x_3 \\ 0 \\ x_3 \end{bmatrix}$$

so the correpsponding eigenvectors will be of the form

$$\mathbf{v}_3 = r \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad , \quad r \in \mathbb{R} - \{0\}$$

3. Find the eigenvalues λ_i, the corresponding eigenvectors $\mathbf{v}_i$ of the following matrices. Also find an invertible matrix $\mathbf{C}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{D} = \mathbf{C}^{-1}\mathbf{AC}$.

(a) $\mathbf{A} = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$

- First, we calculate the eigenvalues and eigenvectors of $\mathbf{A}$.

$$0 = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -3-\lambda & 4 \\ 4 & 3-\lambda \end{vmatrix} = \lambda^2 - 25 = (\lambda - 5)(\lambda + 5) \quad \Rightarrow \quad \lambda = 5, -5$$

The eigenspace corresponding to the eigenvalue $\lambda_1 = 5$ is the null space of

$$\mathbf{A} - (5)\mathbf{I} = \begin{bmatrix} -8 & 4 \\ 4 & -2 \end{bmatrix} \quad \rightsquigarrow \quad \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

or, equivalently, the solution space of

$$\begin{array}{l} 2x_1 - x_2 = 0 \\ 0 = 0 \end{array} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} \in span\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$$

So the eigenspace corresponding to the eigenvalue $\lambda_1 = 5$ is the subspace generated by the vector

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The eigenspace corresponding to the eigenvalue $\lambda_2 = -5$ is the null space of

$$\mathbf{A} - (-5)\mathbf{I} = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \quad \rightsquigarrow \quad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

or, equivalently, the solution space of

$$\begin{array}{l} x_1 + 2x_2 = 0 \\ 0 = 0 \end{array} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} -\frac{1}{2}x_2 \\ x_2 \end{bmatrix} \in span\left(\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right) = span\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)$$

So the eigenspace corresponding to the eigenvalue $\lambda_2 = -5$ is the subspace generated by the vector

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Now that we know the eigenvalues and eigenvectors of $\mathbf{A}$, we can write down the diagonal matrix $\mathbf{D}$ by arranging the eigenvalues of $\mathbf{A}$ along the main diagonal of $\mathbf{D}$

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}$$

The matrix $\mathbf{C}$ can be written down by arranging the eigenvectors of $\mathbf{A}$ (in order) as the column vectors of a 2×2 matrix:

$$\mathbf{C} = [\mathbf{v}_1 \mid \mathbf{v}_2] = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$$

One can easily verify that

$$\mathbf{C}^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & -\frac{2}{5} \end{bmatrix}$$

and that $\mathbf{D} = \mathbf{C}^{-1}\mathbf{AC}$ (however, this fact is already guaranteed by the way we constructed the matrices $\mathbf{D}$ and $\mathbf{C}$).

(b) $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$

- First, we calculate the eigenvalues and eigenvectors of $\mathbf{A}$.

$$0 = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 10 = (\lambda - 2)(\lambda - 5) \quad \Rightarrow \quad \lambda = 2, 5$$

The eigenspace corresponding to the eigenvalue $\lambda_1 = 2$ is the null space of

$$\mathbf{A} - (2)\mathbf{I} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \quad \rightsquigarrow \quad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

or, equivalently, the solution space of

$$\begin{array}{l} x_1 + 2x_2 = 0 \\ 0 = 0 \end{array} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} \in span\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$$

So the eigenspace corresponding to the eigenvalue $\lambda_1 = 2$ is the subspace generated by the vector

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The eigenspace corresponding to the eigenvalue $\lambda_2 = 5$ is the null space of

$$\mathbf{A} - (5)\mathbf{I} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \quad \rightsquigarrow \quad \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

or, equivalently, the solution space of

$$\begin{array}{l} x_1 - x_2 = 0 \\ 0 = 0 \end{array} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} \in span\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

So the eigenspace corresponding to the eigenvalue $\lambda_2 = 5$ is the subspace generated by the vector

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Now that we know the eigenvalues and eigenvectors of $\mathbf{A}$, we can write down the diagonal matrix $\mathbf{D}$ by arranging the eigenvalues of $\mathbf{A}$ along the main diagonal of $\mathbf{D}$

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

The matrix $\mathbf{C}$ can be written down by arranging the eigenvectors of $\mathbf{A}$ (in order) as the column vectors of a 2×2 matrix:

$$\mathbf{C} = [\mathbf{v}_1 \mid \mathbf{v}_2] = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$$

(c) $\mathbf{A} = \begin{bmatrix} 7 & 8 \\ -4 & -5 \end{bmatrix}$

- First, we calculate the eigenvalues and eigenvectors of $\mathbf{A}$.

$$0 = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 7-\lambda & 8 \\ -4 & -5-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) \quad \Rightarrow \quad \lambda = 3, -1$$

The eigenspace corresponding to the eigenvalue $\lambda_1 = 3$ is the null space of

$$\mathbf{A} - (3)\mathbf{I} = \begin{bmatrix} 4 & 8 \\ -4 & -8 \end{bmatrix} \quad \rightsquigarrow \quad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

or, equivalently, the solution space of

$$\begin{array}{l} x_1 + 2x_2 = 0 \\ 0 = 0 \end{array} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} \in span\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$$

So the eigenspace corresponding to the eigenvalue $\lambda_1 = 3$ is the subspace generated by the vector

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The eigenspace corresponding to the eigenvalue $\lambda_2 = -1$ is the null space of

$$\mathbf{A} - (-1)\mathbf{I} = \begin{bmatrix} 8 & 8 \\ -4 & -4 \end{bmatrix} \quad \rightsquigarrow \quad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

or, equivalently, the solution space of

$$\begin{array}{l} x_1 + x_2 = 0 \\ 0 = 0 \end{array} \qquad \Rightarrow \qquad \mathbf{x} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} \in span\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$$

So the eigenspace corresponding to the eigenvalue $\lambda_2 = -1$ is the subspace generated by the vector

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Now that we know the eigenvalues and eigenvectors of $\mathbf{A}$, we can write down the diagonal matrix $\mathbf{D}$ by arranging the eigenvalues of $\mathbf{A}$ along the main diagonal of $\mathbf{D}$

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

The matrix $\mathbf{C}$ can be written down by arranging the eigenvectors of $\mathbf{A}$ (in order) as the column vectors of a 2×2 matrix:

$$\mathbf{C} = [\mathbf{v}_1 \mid \mathbf{v}_2] = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$$

(d) $\mathbf{A} = \begin{bmatrix} 6 & 3 & -3 \\ -2 & -1 & 2 \\ 16 & 8 & -7 \end{bmatrix}$

- The characteristic polynomial of $\mathbf{A}$ is

$$P_{\mathbf{A}}(\lambda) = \begin{vmatrix} 6-\lambda & 3 & -3 \\ -2 & -1-\lambda & 2 \\ 16 & 8 & -7-\lambda \end{vmatrix} = 3\lambda - 2\lambda^2 - \lambda^3 = -\lambda\,(\lambda+3)\,(\lambda-1)$$

So $\mathbf{A}$ has three distinct real eigenvalues: $\lambda_1 = 0$, $\lambda_2 = -3$ and $\lambda_3 = 1$.

The eigenspace corresponding to the first eigenvector $\lambda_1 = 0$ is the null space of

$$\mathbf{A} - (0)\mathbf{I} = \begin{bmatrix} 6 & 3 & -3 \\ -2 & -1 & 2 \\ 16 & 8 & -7 \end{bmatrix} \quad \leftrightsquigarrow \quad \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

is the solution set of

$$\begin{array}{l} x_1 = -\frac{1}{2}x_2 \\ x_3 = 0 \\ 0 = 0 \end{array}$$

So the corresponding eigenvectors are

$$\mathbf{v}_1 \in span\left(\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}\right)$$

The eigenspace corresponding to the eigenvector $\lambda_2 = -3$ is the null space of

$$\mathbf{A} - (2)\mathbf{I} = \begin{bmatrix} 9 & 3 & -3 \\ -2 & 2 & 2 \\ 16 & 8 & -4 \end{bmatrix} \quad \xrightarrow{\text{row reduction}} \quad \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

and so the solution set of

$$\begin{array}{l} x_1 = \frac{1}{2}x_3 \\ x_2 = -\frac{1}{2}x_3 \\ x_3 \text{ is free} \end{array}$$

So the corresponding eigenvectors are

$$\mathbf{v}_2 \in span\left(\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right)$$

The eigenspace corresponding to the first eigenvector $\lambda_3 = 1$ is the null space of

$$\mathbf{A} - (1)\mathbf{I} = \begin{bmatrix} 5 & 3 & -3 \\ -2 & -2 & 2 \\ 16 & 8 & -8 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

is the solution set of

$$\begin{aligned} &x_1 = 0 \\ &x_2 = x_3 \\ &x_3 \text{ is free} \end{aligned}$$

So the corresponding eigenvectors are

$$\mathbf{v}_3 \in span\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$$

From the eigenvalues of $\mathbf{A}$ we can now form the diagonal matrix $\mathbf{D}$:

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And from the coresponding eigenvectors we can form the invertible matrix $\mathbf{C}$

$$\mathbf{C} = [\mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3] = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

such that $\mathbf{D} = \mathbf{C}^{-1}\mathbf{AC}$.

(e) $\mathbf{A} = \begin{bmatrix} -3 & 10 & -6 \\ 0 & 7 & -6 \\ 0 & 0 & 1 \end{bmatrix}$

- The characteristic polynomial of $\mathbf{A}$ is

$$P_{\mathbf{A}}(\lambda) = \begin{vmatrix} -3-\lambda & 10 & -6 \\ 0 & 7-\lambda & -6 \\ 0 & 0 & 1-\lambda \end{vmatrix} = -(\lambda+3)(\lambda-7)(\lambda-1)$$

So $\mathbf{A}$ has three distinct real eigenvalues: $\lambda_1 = -3$, $\lambda_2 = 7$ and $\lambda_3 = 1$.

The eigenspace corresponding to the first eigenvector $\lambda_1 = 0$ is the null space of

$$\mathbf{A} - (-3)\mathbf{I} = \begin{bmatrix} 0 & 10 & -6 \\ 0 & 10 & -6 \\ 0 & 0 & 4 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

is the solution set of

$$\begin{aligned} x_2 &= 0 \\ x_3 &= 0 \\ 0 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} &x_1 \text{ is free} \\ &x_2 = 0 \\ &x_3 = 0 \end{aligned}$$

So the corresponding eigenvectors are

$$\mathbf{v}_1 \in span\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

The eigenspace corresponding to the eigenvector $\lambda_2 = 7$ is the null space of

$$\mathbf{A} - (7)\mathbf{I} = \begin{bmatrix} -10 & 10 & -6 \\ 0 & 0 & -6 \\ 0 & 0 & -6 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

is the solution set of

$$\begin{array}{lcl} x_1 - x_2 = 0 & & x_1 = x_2 \\ x_3 = 0 & \Rightarrow & x_2 \text{ is free} \\ 0 = 0 & & x_3 = 0 \end{array}$$

So the corresponding eigenvectors are

$$\mathbf{v}_2 \in span\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)$$

The eigenspace corresponding to the first eigenvector $\lambda_3 = 1$ is the null space of

$$\mathbf{A} - (1)\mathbf{I} = \begin{bmatrix} -4 & 10 & -6 \\ 0 & 6 & -6 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

is the solution set of

$$\begin{array}{lcl} x_1 - x_3 = 0 & & x_1 = x_3 \\ x_2 - x_3 = 0 & \Rightarrow & x_2 = x_3 \\ 0 = 0 & & x_3 \text{ is free} \end{array}$$

So the corresponding eigenvectors are

$$\mathbf{v}_3 \in span\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$$

From the eigenvalues of $\mathbf{A}$ we can now form the diagonal matrix $\mathbf{D}$:

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And from the coresponding eigenvectors we can form the invertible matrix $\mathbf{C}$

$$\mathbf{C} = [\mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

such that $\mathbf{D} = \mathbf{C}^{-1}\mathbf{AC}$.

4. Determine whether or not the following matrices are diagonalizable.

(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 0 & -4 \\ 6 & -4 & 3 \end{bmatrix}$

- Yes, because the matrix is real and symmetric. (See Theorem 14.8 in the Lecture 14.)

(b) $\mathbf{A} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

- The matrix $\mathbf{A} - \lambda\mathbf{I}$ is upper triangular and so its determinant is readily computed. We have

$$p_{\mathbf{A}}(\lambda) = det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 3)(\lambda - 2)(\lambda - 1)$$

and so $\mathbf{A}$ has three distinct eigenvalues: $\lambda = 3, 2, 1$. This implies that $\mathbf{A}$ has at least three linearly independent eigenvectors, and that is that is needed for a 3×3 matrix to be diagonalizable. Hence, $\mathbf{A}$ is diagonalizable.

(c) $\mathbf{A} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

- Let us calculate the characteristic polynomial of $\mathbf{A}$:

$$P_{\mathbf{A}}(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^3$$

We thus have only one eigenvalue, $\lambda = 3$. The corresponding eigenspace is the null space of

$$\mathbf{A} - (3)\mathbf{I} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

or, equivalently, the solution space of

$$\begin{array}{l} x_2 = 0 \\ x_3 = 0 \\ 0 = 0 \end{array} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} \in span\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

So the eigenspace is just 1-dimensional. But we need three linearly independent eigenvectors to construct the matrix $\mathbf{C}$ that diagonalizes $\mathbf{A}$. Hence, $\mathbf{A}$ is not diagonalizable. (Remark: from the row echelon form of $\mathbf{A} - (3)\mathbf{I}$ it was already apparent that there would be only one linearly independent eigenvalues. In general, if a row echelon form of $\mathbf{A} - \lambda\mathbf{I}$ has k columns without pivots, then $\mathbf{A}$ will have exactly k linearly independent eigenvectors with eigenvalue λ. For as we have seen, counting the columns without pivots in a REF of a matrix $\mathbf{M}$ reveals the dimension of the solution set of $\mathbf{Mx} = \mathbf{0}$.)