Math 3013
Solutions to Homework Set 7

1. Show by direct calculation that

$$\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = -\det\begin{pmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

(This demonstrates how the determinant behaves after a row interchange).

- We'll use cofactor expansions:

$$\det(LHS) = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$\begin{aligned} \det(RHS) &= a_1(c_2b_3 - c_3b_2) - a_2(c_1b_3 - c_3b_1) + a_3(c_1b_2 - c_2b_1) \\ &= a_1(-b_2c_3 + b_3c_2) - a_2(-b_1c_3 + b_3c_1) + a_3(-b_1c_2 + b_2c_1) \\ &= -a_1(b_2c_3 - b_3c_2) + a_2(b_1c_3 - b_3c_1) - a_3(b_1c_2 - b_2c_1) \\ &= -\det(LHS) \end{aligned}$$

: $a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$

2. Compute the determinants of the following matrices.

(a) $\mathbf{A} = \begin{bmatrix} 2 & 3 & -1 \\ 5 & -7 & 1 \\ -3 & 2 & -1 \end{bmatrix}$

- Instead of row reducing $\mathbf{A}$ we'll row reduce $\mathbf{A}^T$ because that turns out to be a bit easier, and is legitimate since $\det(\mathbf{A}^T) = \det(\mathbf{A})$.

$$\begin{aligned} \det(\mathbf{A}) &= \det(\mathbf{A}^T) = \det\begin{bmatrix} 2 & 5 & -3 \\ 3 & -7 & 2 \\ -1 & 1 & -1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} = -\det\begin{bmatrix} -1 & 1 & -1 \\ 3 & -7 & 2 \\ 2 & 5 & -3 \end{bmatrix} \\ \xrightarrow[R_3 \to R_3 + 2R_1]{R_2 \to R_2 + 3R_1} &= -\det\begin{bmatrix} -1 & 1 & -1 \\ 0 & -4 & -1 \\ 0 & 7 & -5 \end{bmatrix} \xrightarrow{R_3 \to R_3 + \frac{7}{4}R_2} = -\det\begin{bmatrix} -1 & 1 & -1 \\ 0 & -4 & -1 \\ 0 & 0 & -\frac{27}{4} \end{bmatrix} \\ &= -(-1)(-4)\left(-\frac{27}{4}\right) = 27 \end{aligned}$$

(b) $\mathbf{A} = \begin{bmatrix} 5 & 2 & 4 & 0 \\ 2 & -3 & -1 & 2 \\ 3 & -4 & 3 & 7 \\ 1 & -1 & 0 & 1 \end{bmatrix}$

- We'll use row reduction.

$$
\begin{aligned}
\det\begin{pmatrix} 5 & 2 & 4 & 0 \\ 2 & -3 & -1 & 2 \\ 3 & -4 & 3 & 7 \\ 1 & -1 & 0 & 1 \end{pmatrix} &= -\det\begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & -3 & -1 & 2 \\ 3 & -4 & 3 & 7 \\ 5 & 2 & 4 & 0 \end{pmatrix} \\
&= -\det\begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & 3 & 4 \\ 0 & 7 & 4 & -5 \end{pmatrix} = -\det\begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & -3 & -5 \end{pmatrix} \\
&= -4\det\begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -3 & -5 \end{pmatrix} = -4\det\begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2 \end{pmatrix} \\
&= -8
\end{aligned}
$$

(c) $\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 3 & -1 & 2 & 0 & 0 \\ 0 & 4 & 1 & -1 & 2 \\ 0 & 0 & -3 & 2 & 4 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix}$

- We'll start out using row reduction

$$
\begin{aligned}
\det\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 3 & -1 & 2 & 0 & 0 \\ 0 & 4 & 1 & -1 & 2 \\ 0 & 0 & -3 & 2 & 4 \\ 0 & 0 & 0 & -1 & 3 \end{pmatrix} &= \det\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & -\frac{5}{2} & 2 & 0 & 0 \\ 0 & 4 & 1 & -1 & 2 \\ 0 & 0 & -3 & 2 & 4 \\ 0 & 0 & 0 & -1 & 3 \end{pmatrix} \\
&= \det\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & -\frac{5}{2} & 2 & 0 & 0 \\ 0 & 0 & \frac{21}{5} & -1 & 2 \\ 0 & 0 & -3 & 2 & 4 \\ 0 & 0 & 0 & -1 & 3 \end{pmatrix} \\
&= (2)\left(\left(-\frac{5}{2}\right)\det\begin{pmatrix} \frac{21}{5} & -1 & 2 \\ -3 & 2 & 4 \\ 0 & -1 & 3 \end{pmatrix} - 2\det\begin{pmatrix} 0 & -1 & 2 \\ 0 & 2 & 4 \\ 0 & -1 & 3 \end{pmatrix} + 0 + 0\right) \\
&= (2)\left(-\frac{5}{2}\right)\left(\frac{21}{5}\left((2)(3)-(4)(-1)\right) - (-3)\left((-1)(3)-(2)(-1)\right) + (0)\left((-1)(4)-(2)(2)\right)\right) \\
&= (2)\left(-\frac{5}{2}\right)(42-3+0) \\
&= -195
\end{aligned}
$$

(d) $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 2 & 0 & -3 \\ 0 & -2 & 1 & 0 & 0 \\ 5 & -3 & 2 & 0 & 0 \\ -3 & 4 & 0 & 0 & 0 \end{bmatrix}$

- We'll carry out a cofactor expansion along the fourth column.

$$
\begin{aligned}
\det\begin{pmatrix} 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 2 & 0 & -3 \\ 0 & -2 & 1 & 0 & 0 \\ 5 & -3 & 2 & 0 & 0 \\ -3 & 4 & 0 & 0 & 0 \end{pmatrix} &= -(3)\det\begin{pmatrix} 0 & 0 & 2 & -3 \\ 0 & -2 & 1 & 0 \\ 5 & -3 & 2 & 0 \\ -3 & 4 & 0 & 0 \end{pmatrix} + 0 - 0 + 0 - 0 \\
&= -(3)\left(-(-3)\det\begin{pmatrix} 0 & -2 & 1 \\ 5 & -3 & 2 \\ -3 & 4 & 0 \end{pmatrix} + 0 - 0 + 0\right) \\
&= (3)(-3)(0 - (-2)(0+6) + (1)(20-9)) \\
&= -207
\end{aligned}
$$

(e) $\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 \\ -5 & 2 & 6 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -2 & 8 \end{bmatrix}$

- This matrix is block diagonal. To compute its determinant we can compute the determinants of the diagonal blocks and multiply them together.

$$
\begin{aligned}
\det\begin{pmatrix} 2 & -1 & 3 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 \\ -5 & 2 & 6 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -2 & 8 \end{pmatrix} &= \det\begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & 4 \\ -5 & 2 & 6 \end{pmatrix}\det\begin{pmatrix} 1 & 4 \\ -2 & 8 \end{pmatrix} \\
&= (0 + (1)(12+15) - (4)(4-5)) * (8+8) \\
&= 496
\end{aligned}
$$

3. For each matrix $\mathbf{A}$ below, let $\mathbf{C}_A$ be the cofactor matrix $(\mathbf{C}_A)_{ij} = (-1)^{i+j}\det\left(A^{(i,j)}\right)$. Use the formula $\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}(\mathbf{C}_A)^T$ to compute $\mathbf{A}^{-1}$.

(a) $\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$

- We have $\det(\mathbf{A}) = 4 - 2 = 2$. We now compute the cofactor of $\mathbf{A}$.

$$
\left.\begin{aligned}
C_{11} &= (-1)^{1+1}\det(1) &&= 1 \\
C_{12} &= (-1)^{1+2}\det(2) &&= -2 \\
C_{21} &= (-1)^{2+1}\det(1) &&= -1 \\
C_{22} &= (-1)^{2+2}\det(4) && 4
\end{aligned}\right\} \Rightarrow \mathbf{C} = \begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix} \quad \Rightarrow \mathbf{C}^T = \begin{bmatrix} 1 & -1 \\ -2 & 4 \end{bmatrix}
$$

We now can apply the formula

$$
\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^T = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -1 & 2 \end{bmatrix}
$$

(b) $\mathbf{A} = \begin{bmatrix} 3 & 0 & 4 \\ -2 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

- We have $\det(\mathbf{A}) = (3)(2-1) - (0)(-4-3) + 4(-2-3) = -17$. The cofactors are

$$
\begin{array}{lll}
C_{11} = \det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = 1 & C_{12} = -\det \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix} = -7 & C_{13} = \det \begin{bmatrix} -2 & 1 \\ 3 & 1 \end{bmatrix} = -5 \\
C_{21} = -\det \begin{bmatrix} 0 & 4 \\ 1 & 2 \end{bmatrix} = 4 & C_{22} = \det \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix} = -6 & C_{23} = -\det \begin{bmatrix} 3 & 0 \\ 3 & 1 \end{bmatrix} = -3 \\
C_{31} = \det \begin{bmatrix} 0 & 4 \\ 1 & 1 \end{bmatrix} = -4 & C_{32} = -\det \begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix} = -11 & C_{33} = \det \begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix} = 3
\end{array}
$$

$$
\Rightarrow \quad \mathbf{C} = \begin{bmatrix} 1 & -7 & -5 \\ 4 & -6 & -3 \\ -4 & -11 & 3 \end{bmatrix} \quad \Rightarrow \quad \mathbf{C}^T = \begin{bmatrix} 1 & 4 & -4 \\ -7 & -6 & -11 \\ -5 & -3 & 3 \end{bmatrix}
$$

$$
\Rightarrow \quad \mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^T = \frac{-1}{17}\begin{bmatrix} 1 & 4 & -4 \\ -7 & -6 & -11 \\ -5 & -3 & 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{17} & -\frac{4}{17} & \frac{4}{17} \\ -\frac{7}{17} & \frac{6}{17} & \frac{11}{17} \\ \frac{5}{17} & \frac{3}{17} & -\frac{3}{17} \end{bmatrix}
$$

4. Solve the following systems of linear equations using Cramers's Rule.

(a)
$$
\begin{aligned} x_1 - 2x_2 &= 1 \\ 3x_1 + 4x_2 &= 3 \end{aligned}
$$

- For this problem we have

$$
\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \Rightarrow \quad \mathbf{B}_1 = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}
$$

Crammer's Rule tells us that the solution $\mathbf{x} = (x_1, x_2)$ of the above linear system is given by

$$
\begin{aligned}
x_1 &= \frac{\det(\mathbf{B}_1)}{\det(\mathbf{A})} = \frac{4+6}{4+6} = 1 \\
x_2 &= \frac{\det(\mathbf{B}_2)}{\det(\mathbf{A})} = \frac{3-3}{4+6} = 0
\end{aligned}
$$

(b)
$$
\begin{aligned} x_1 + 2x_2 - x_3 &= -2 \\ 2x_1 + x_2 + x_3 &= 0 \\ 3x_1 - x_2 + 5x_3 &= 1 \end{aligned}
$$

- For this problem we have

$$
\begin{aligned}
\mathbf{A} &= \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 3 & -1 & 5 \end{bmatrix} & \Rightarrow \quad \det(\mathbf{A}) = -3 \\
\mathbf{B}_1 &= \begin{bmatrix} -2 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 5 \end{bmatrix} & \Rightarrow \quad \det(\mathbf{B}_1) = -9 \\
\mathbf{B}_2 &= \begin{bmatrix} 1 & -2 & -1 \\ 2 & 0 & 1 \\ 3 & 1 & 5 \end{bmatrix} & \Rightarrow \quad \det(\mathbf{B}_2) = 11 \\
\mathbf{B}_3 &= \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} & \Rightarrow \quad \det(\mathbf{B}_3) = 7
\end{aligned}
$$

So, by Crammer's Rule, the solution ot the linear system is given by

$$\begin{aligned} x_1 &= \frac{\det(\mathbf{B}_1)}{\det(\mathbf{A})} = \frac{-9}{-3} = 3 \\ x_2 &= \frac{\det(\mathbf{B}_2)}{\det(\mathbf{A})} = -\frac{11}{3} \\ x_3 &= \frac{\det(\mathbf{B}_3)}{\det(\mathbf{A})} = -\frac{7}{3} \end{aligned}$$