1. Compute the determinants of the following matrices.

(a) \[
\begin{pmatrix}
1 & 3 \\
-1 & 2
\end{pmatrix}
\]

\[\det \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix} = (1)(2) - (3)(-1) = 5 \]

(b) \[
\begin{pmatrix}
1 & 0 & -1 \\
3 & 2 & 1 \\
-1 & 1 & 0
\end{pmatrix}
\]

\[\det \begin{pmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \\ -1 & 1 & 0 \end{pmatrix} = (1)(-1)^{3+1} \det \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 3 & 1 & 4 \end{pmatrix} + (0)(-1)^{3+2} \det \begin{pmatrix} 3 & 1 & 0 \\ -1 & 0 & 1 \\ 3 & 1 & 4 \end{pmatrix} + (-1)(-1)^{3+3} \det \begin{pmatrix} 3 & 2 & 0 \\ -1 & 1 & 0 \\ 3 & 1 & 4 \end{pmatrix} = -6 \]

2. Use a cofactor expansion to compute the determinant of \( A = \begin{pmatrix}
1 & 0 & 0 & 0 \\
3 & 2 & 0 & 0 \\
-1 & 1 & 3 & 0 \\
2 & 3 & 1 & 4
\end{pmatrix}\)

- Cofactor expansions along the first row:

\[\det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ -1 & 1 & 3 & 0 \\ 2 & 3 & 1 & 4 \end{pmatrix} = (1)(-1)^{3+1} \det \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 3 & 1 & 4 \end{pmatrix} + 0 + 0 + 0 \]

\[= \det \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 3 & 1 & 4 \end{pmatrix} = (2)(-1)^{1+1} \det \begin{pmatrix} 3 & 0 \\ 1 & 4 \end{pmatrix} \]

\[= 2 \det \begin{pmatrix} 3 & 0 \\ 1 & 4 \end{pmatrix} \]

\[= 2 \left( 3(1) - 0 \cdot 1 \right) = 2 \cdot \det \begin{pmatrix} 3 & 0 \\ 1 & 4 \end{pmatrix} \]

\[= (2)(3)(4) = 24 \]

3. Use row reduction to compute the determinant of \( A = \begin{pmatrix}
1 & 1 & -1 & 2 \\
2 & 1 & 0 & 1 \\
3 & 2 & -1 & 3 \\
2 & 1 & 1 & 2
\end{pmatrix}\)
4. Use the result of Problem 1(b) to determine if the vectors \( \mathbf{a} = [1, 0, -1] \), \( \mathbf{b} = [3, 2, 1] \) and \( \mathbf{c} = [-1, 1, 0] \) are linearly independent.

- If the vectors \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \) are linearly independent, then the matrix \( \mathbf{A} \) constructed by using \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \) as rows (or columns) must have a \textit{nonzero} determinant. The matrix in problem 1 (b) is has the vectors \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \) as its rows and this matrix has determinant \(-6\) which is not zero. Hence, the vectors \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \) are linearly independent.

5. Use the result of Problem 3 to determine if the linear system
\[
\begin{align*}
    x_1 + x_2 - x_3 + 2x_4 &= 0 \\
    2x_1 + x_2 + x_4 &= 0 \\
    3x_1 + 2x_2 - x_3 + 3x_4 &= 0 \\
    2x_1 + x_2 + x_3 + 2x_4 &= 0
\end{align*}
\]
has a unique solution.

- An \( n \times n \) linear system \( \mathbf{A}\mathbf{x} = \mathbf{b} \) has a unique solution if and only if the determinant of the coefficient matrix is non-zero. The matrix in Problem 3 is the coefficient matrix for this linear system and its determinant is equal to 0. Therefore, this linear system does not have a unique solution.

6. Use Cramer’s Rule to determine the solution (if any) of
\[
\begin{align*}
    x_1 + 2x_2 + x_3 &= 2 \\
    2x_1 - 2x_2 &= 4 \\
    x_2 + x_3 &= 3
\end{align*}
\]

- For this linear system we have
\[
\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & -2 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}
\]

We form three associated matrices \( \mathbf{B}_i, i = 1, 2, 3 \), by replacing the \( i^{th} \) column of \( \mathbf{A} \) with \( \mathbf{b} \).
\[
\mathbf{B}_1 = \begin{pmatrix} 2 & 2 & 1 \\ 4 & -2 & 0 \\ 3 & 1 & 1 \end{pmatrix}, \quad \mathbf{B}_2 = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 0 & 3 & 1 \end{pmatrix}, \quad \mathbf{B}_3 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 4 \\ 0 & 1 & 3 \end{pmatrix}
\]
We have
\[ \det(A) = -4 \]
\[ \det(B_1) = -2 \]
\[ \det(B_2) = 6 \]
\[ \det(B_3) = -18 \]

Cramer’s Rule says the components \( x_1, x_2, x_3 \) of the solution vector are given by \( x_i = \frac{\det(B_i)}{\det(A)} \). Thus,
\[ x_1 = \frac{-2}{-4} = \frac{1}{2} \]
\[ x_2 = \frac{6}{-4} = -\frac{3}{2} \]
\[ x_3 = \frac{-18}{-4} = \frac{9}{2} \]

7. Compute the cofactor \( C \) of matrix of \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) and use the formula
\[ A^{-1} = \frac{1}{\det(A)} C^T \]
to get a general formula for the inverse of a \( 2 \times 2 \) matrix.

- First, we construct the cofactor matrix, \( C \). Its entries are given by
\[ (C)_{ij} = (-1)^{i+j} \det(M_{ij}) \]
where \( M_{ij} \) is the matrix obtained from \( A \) by removing its \( i^{th} \) row and \( j^{th} \) column. Thus,
\[ C_{11} = (-1)^{1+1} \det([d]) \quad C_{12} = (-1)^{1+2} \det([c]) \]
\[ C_{21} = (-1)^{2+1} \det([b]) \quad C_{22} = (-1)^{2+2} \det([a]) \]
or
\[ C = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \]

We also have
\[ \det(A) = \det\left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = ad - bc \]
and the transpose of \( C \) is
\[ C^T = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \]
Thus,
\[ A^{-1} = \frac{1}{\det(A)} C^T = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} \]