Math 3013
Solutions to Problem Set 6

1. Consider the set $\mathcal{F}(\mathbb{R})$ of functions on the real line. This can be given the structure of a vector space by setting

$$\begin{aligned}
(f+g)(x) &\equiv f(x)+g(x) &&\text{(vector addition of functions)}\\
(\lambda f)(x) &\equiv \lambda f(x) &&\text{(scalar multiplication of functions)}\\
\mathbf{0}_{\mathcal{F}(\mathbb{R})}(x) &= 0 \text{ for all } x &&\text{(the zero vector in } \mathcal{F}(\mathbb{R})\text{)}
\end{aligned}$$

Determine if the following sets are subspaces of $\mathcal{F}(\mathbb{R})$:

(a) $S = \left\{ f \in \mathcal{F}(\mathbb{R}) \mid \frac{df}{dx} = 0 \right\}$

- Suppose $f, g \in S$. It suffices to check that $\alpha f + \beta g$ is also in S for all real numbers α, β. The condition for a function to be in S is that its derivative must vanish. Thus we check

$$\begin{aligned}
0 &\stackrel{?}{=} \frac{d}{dx}(\alpha f(x) + \beta g(x))\\
&= \frac{d}{dx}(\alpha f(x)) + \frac{d}{dx}(\beta g(x)) &&\text{(differentiating term by term)}\\
&= \alpha \frac{df}{dx}(x) + \beta \frac{dg}{dx}(x) &&\text{(differentiation ignores constants)}\\
&= \alpha \cdot 0 + \beta \cdot 0 &&\text{(because } f, g \in S\text{)}\\
&= 0
\end{aligned}$$

So, yes, S is a subspace of $\mathcal{F}(\mathbb{R})$.

(b) $T = \left\{ f \in \mathcal{F}(\mathbb{R}) \mid \int_0^1 f(x)\,dx = 0 \right\}$

- This is very similar to part (a), since we can integrate term by term and because integration ignores constants. Suppose $f, g \in T$ and consider the linear combination $\alpha f + \beta g$. Then

$$\begin{aligned}
0 &\stackrel{?}{=} \int_0^1 (\alpha f(x) + \beta g(x))\,dx\\
&= \alpha \int_0^1 f(x)\,dx \quad + \quad \beta \int_0^1 g(x)\,dx\\
&= \alpha \cdot 0 + \beta \cdot 0\\
&= 0
\end{aligned}$$

So $\alpha f + \beta g \in T$, and since T is closed under arbitrary linear combinations, T is a subspace of $\mathcal{F}(\mathbb{R})$.

(c) $U = \{ f \in \mathcal{F}(\mathbb{R}) \mid f(3) = 0 \}$

- Let $f, g \in U$ and consider $\alpha f + \beta g$. We check

$$\begin{aligned}
0 &\stackrel{?}{=} (\alpha f + \beta g)(3)\\
&= \alpha f(3) + \beta g(3)\\
&= \alpha \cdot 0 + \beta \cdot 0\\
&= 0
\end{aligned}$$

So U is closed under arbitrary linear combinations, and so U is a subspace of $\mathcal{F}(\mathbb{R})$.

(d) $V = \{ f \in \mathcal{F}(\mathbb{R}) \mid f(0) = 3 \}$

- V will not be closed under vector addition or scalar multiplication. Consider $f \in V$ then consider is scalar multiple by 0. Then

$$\begin{aligned}(0 \cdot f)(0) &= 0 \cdot f(0) = 0 \cdot 3 = 0 \\ &\neq 3\end{aligned}$$

So $(0 \cdot f)$ is not in V, and so V is **not** closed under scalar multiplication; hence V is **not** a subspace of $\mathcal{F}(\mathbb{R})$.

Alternatively, consider $f, g \in V$ and their sum $f + g$. We have

$$\begin{aligned}(f+g)(0) &= f(0) + g(0) = 3 + 3 = 6 \\ &\neq 3\end{aligned}$$

and so V is not closed under vector addition, and so V is **not** a subspace of $\mathcal{F}(\mathbb{R})$.

2. Let P_2 be the vector space of polynomials of degree ≤ 2.

(a) Show that $B = \{1, x, x^2\}$ is a basis for P_2

- We need to check two things. First, that $P_2 = span(1, x, x^2)$ and, secondly, that $\{1, x, x^2\}$ are linearly independent. By the very definition of P_2

$$\begin{aligned}P_2 &\equiv \{a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\} \\ &= \{a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^1 \mid a_0, a_1, a_2 \in \mathbb{R}\} \\ &\equiv span(1, x, x^2)\end{aligned}$$

the first condition follows. To see if $\{1, x, x^2\}$ are linearly dependent we look for non-trivial solutions of

(*) $$a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \qquad \text{(the zero polynomial)}$$

But two polynomials are equal only if their coefficients coincide; hence (*) requires

$$a_0 = 0 \quad , \quad a_1 = 0 \quad , \quad a_1 = 0$$

Thus, $\{1, x, x^2\}$ are linearly independent.

(b) What are the coordinates of a polynomial $2 + x^2$ with respect to B.

- We use the coordinatization map

$$i_B : a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2 \quad \longmapsto \quad [a_0, a_1, a_2] \in \mathbb{R}^3$$

We then have

$$2 + x^2 \quad \longmapsto \quad [2, 0, 1]$$

(c) Show that $T : P_2 \to P_2 : p \longmapsto x\frac{dp}{dx} - p$ is a linear transformation.

- Let $p_1, p_2 \in P_2$, $\alpha, \beta \in \mathbb{R}$ and consider the linear combination $\alpha p_1 + \beta p_2$. Then

$$\begin{aligned}T(\alpha p_1 + \beta p_2) &= x\frac{d}{dx}(\alpha p_1 + \beta p_2) - (\alpha p_1 + \beta p_2) \\ &= \alpha x \frac{dp_1}{dx} + \beta x \frac{dp_2}{dx} - \alpha p_1 + \beta p_2 \\ &= \alpha\left(x\frac{dp_1}{dx} - p_1\right) + \beta\left(x\frac{dp_2}{dx} - p_2\right) \\ &= \alpha T(p_1) + \beta T(p_2)\end{aligned}$$

and so T is a linear transformation.

(d) Find the matrix $\mathbf{A}_{T,B,B}$ representing T.

- The matrix $\mathbf{A}_{T,B,B}$ is formed by applying T to each basis vector for P_2, converting these results to their respective coordinate vectors and using these coordinate vectors to form the columns of $\mathbf{A}_{T,B,B}$.

$$\begin{array}{ccccccc} 1 & \xrightarrow{T} & x\dfrac{d}{dx}(1) - 1 & = & -1 = (-1)\cdot 1 + (0)\cdot x + (0)\cdot x^2 & \longmapsto & [-1,0,0] \\ x & \xrightarrow{T} & x\dfrac{d}{dx}(x) - x & = & 0 = (0)\cdot 1 + (0)\cdot x + (0)\cdot x^2 & \longmapsto & [0,0,0] \\ x^2 & \xrightarrow{T} & x\dfrac{d}{dx}\left(x^2\right) - x^2 & = & x^2 = (0)\cdot 1 + (0)\cdot x + (1)\cdot x^2 & \longmapsto & [0,0,1] \end{array}$$

Thus,

$$\mathbf{A}_{T,B,B} = \left[\left[\begin{array}{c} -1 \\ 0 \\ 0 \end{array}\right]\left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]\left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]\right] = \left[\begin{array}{ccc} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

(e) What is the kernel of T?

- We'll find the kernel of T by finding the null space of $\mathbf{A}_{T,B,B}$ (the solution set of $\mathbf{A}_{T,B,B}\mathbf{x} = \mathbf{0}$) and then mapping the corresponding coordinate vectors back to polynomials. Now $\mathbf{A}_{T,B,B}$ trivially row reduces to

$$\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right) \quad \Rightarrow \quad x_1 = 0\ ,\ x_2 \text{ is free}\ ,\ x_3 = 0$$

$$\Rightarrow \quad NullSp\left(\mathbf{A}_{T,B,B}\right) = span\left(\left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right]\right)$$

Since $[0,1,0] \longleftrightarrow$ the polynomial x, we conclude

$$\ker(T) = span\,(x) \quad .$$

3. Let P be the vector space of polynomials. Prove that $span\,(1,x) = span\,(1+2x,x)$.

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$$span\,(1,x) = \{a_0 + a_1 x \mid a_0, a_1 \in \mathbb{R}\} = \text{polynomials of degree } \leq 2$$

On the other hand,

$$\begin{array}{rcl} span\,(1+2x,x) & = & \{b_0(1+2x) + b_1 x \mid b_0, b_1 \in \mathbb{R}\} \\ & = & \{b_0 + (b_1 + 2b_0)\,x \mid b_0, b_1 \in \mathbb{R}\} \\ & \subset & span\,(1,x) \end{array}$$

Note also that the polynomials $1+2x$ and x are linearly independent: for

$$b_0(1+2x) + b_1 x = 0 \quad \Rightarrow \quad b_0 = 0 \quad , \quad b_1 = 0$$

Therefore, $span\,(1+2x,x)$ is a subspace of $span\,(1,x)$ and it has the same dimension as $span\,(1,x)$; so it must coincide with $span\,(1,x)$.

4. Consider the following set of polynomials

$$\left\{1\ ,\ 4x+3\ ,\ 3x-4\ ,\ x^2+2\ ,\ x-x^2\right\}$$

(a) Determine if these polynomials are linearly independent in P_2.

- Let's adopt the usual basis $\{1, x, x^2\}$ for polynomials of degree ≤ 2 and map these polynomials to their coordinate vectors:

$$\begin{array}{rcl} 1 & \to & [1,0,0] \\ 4x+3 & \to & [3,4,0] \\ [3x-4] & \to & [-4,3,0] \\ x^2+2 & \to & [2,0,1] \\ x-x^2 & \to & [0,1,-1] \end{array}$$

  Next, we transform the question about the original polynomials to a question about the vectors just found:

  Are the vectors $[1,0,0]$, $[3,4,0]$, $[-4,3,0]$, $[2,0,1]$, $[0,1,-1]$ linearly independent?

  To answer this question, we arrange the vectors as the rows of a matrix and row reduce to row echelon form

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 4 & 0 \\ -4 & 3 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

  Observing the zero rows in the row echelon form, we conclude first that the vectors $[1,0,0], \ldots, [0,1,-1]$ are **not** linearly independent. And then we can conclude that the original polynomials can not be linearly independent either.

(b) What is the dimension of

$$S = span\left(1\ ,\ 4x+3\ ,\ 3x-4\ ,\ x^2+2\ ,\ x-x^2\right)$$

- Continuing to use the coordinate vectors $[1,0,0], \cdots, [0,1,-1]$ to study the polynomials $1, \ldots, x-x^2$, we observe that since the row echelon form of the matrix formed by using $[1,0,0], \ldots, [0,1,-1]$ as rows has 3 non-zero rows the span of the row vectors is 3-dimensional. Hence, the span of the original set of polynomials is 3-dimensional.