1. Determine which of the following mappings are linear transformations.

(a) $T : \mathbb{R}^3 \to \mathbb{R}^2 : T ([x_1, x_2, x_3]) = [x_1 + x_2, x_1 - 3x_2]$

- This mapping is linear since if $v = [x_1, x_2, x_3]$
  
  $T (\lambda v) = T (\lambda [x_1, x_2, x_3])$
  
  $= T ([\lambda x_1, \lambda x_2, \lambda x_3])$
  
  $= [\lambda x_1 + \lambda x_2, \lambda x_1 - 3\lambda x_2]$
  
  $= \lambda [x_1 + x_2, x_1 - 3x_2]$
  
  $= \lambda T ([x_1, x_2, x_3])$
  
  $= \lambda T (v)$  \( (T \text{ preserves scalar multiplication}) \)

and if $v = [x_1, x_2, x_3]$ and $v' = [x'_1, x'_2, x'_3]$

$T (v + v') = T ([x_1 + x'_1, x_2 + x'_2, x_3 + x'_3])$

$= [x_1 + x'_1 + x_2 + x'_2, x_1 + x'_1 - 3(x_2 + x'_2)]$

$= [x_1 + x_2, x_1 - 3x_2] + [x'_1 + x'_2, x'_1 - 3x'_2]$

$= T (v) + T (v')$  \( (T \text{ preserves vector addition}) \)

(b) $T : \mathbb{R}^3 \to \mathbb{R}^4 : T ([x_1, x_2, x_3]) = [0, 0, 0, 0]$

- This mapping is linear since if $v = [x_1, x_2, x_3]$
  
  $T (\lambda v) = T ([\lambda x_1, \lambda x_2, \lambda x_3])$
  
  $= [0, 0, 0, 0]$
  
  $= \lambda [0, 0, 0, 0]$
  
  $= \lambda T ([x_1, x_2, x_3])$
  
  $= \lambda T (v)$  \( (T \text{ preserves scalar multiplication}) \)

and if $v = [x_1, x_2, x_3]$ and $v' = [x'_1, x'_2, x'_3]$

$T (v + v') = T ([x_1 + x'_1, x_2 + x'_2, x_3 + x'_3])$

$= [0, 0, 0, 0]$

$= [0, 0, 0, 0] + [0, 0, 0, 0]$

$= T (v) + T (v')$  \( (T \text{ preserves vector addition}) \)

(c) $T : \mathbb{R}^3 \to \mathbb{R}^4 : T ([x_1, x_2, x_3]) = [1, 1, 1, 1]$

- This mapping is not linear since if $v = [x_1, x_2, x_3]$
  
  $T (v) = [1, 1, 1, 1]$
  
  $T (2v) = [1, 1, 1, 1] \neq 2[1, 1, 1, 1] = 2T (v)$

So the mapping does not preserve scalar multiplication.

(d) $T : \mathbb{R}^2 \to \mathbb{R}^3 : T ([x_1, x_2]) = [x_1 - x_2, x_2 + 1, 3x_1 - 2x_2]$
• This mapping is not linear since, e.g., if \( \mathbf{v} = [1, 1] \)

\[
T(\mathbf{v}) = [0, 2, 1] \\
T(2\mathbf{v}) = T([2, 2]) = [0, 3, 2] \neq [0, 4, 2] = 2T(\mathbf{v})
\]

So the mapping does not preserve scalar multiplication.

2. For each of the following, assume \( T \) is a linear transformation, from the data given, compute the specified value.

(a) Given \( T([1, 0]) = [3, -1] \), and \( T([0, 1]) = [-2, 5] \), find \( T([4, -6]) \).

• Because linear transformations preserve scalar multiplication and vector addition, they also preserve linear combinations:

\[
T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)
\]

Now take \( \mathbf{e}_1 = [1, 0] \) and \( \mathbf{e}_2 = [0, 1] \). Then

\[
T([4, -6]) = T(4\mathbf{e}_1 - 6\mathbf{e}_2) \\
= 4T(\mathbf{e}_1) - 6T(\mathbf{e}_2) \\
= 4[3, -1] - 6[-2, 5] \\
= [12 + 12, -4 - 30] \\
= [24, -34]
\]

(b) Given \( T([1, 0, 0]) = [3, 1, 2] \), \( T([0, 1, 0]) = [2, -1, 4] \), and \( T([0, 0, 1]) = [6, 0, 1] \), find \( T([2, -5, 1]) \).

• As in Part (a), we set \( \mathbf{e}_1 = [1, 0, 0] \), \( \mathbf{e}_2 = [0, 1, 0] \), and \( \mathbf{e}_3 = [0, 0, 1] \) and then compute

\[
T([2, -5, 1]) = T(2\mathbf{e}_1 - 5\mathbf{e}_2 + \mathbf{e}_3) \\
= 2T(\mathbf{e}_1) - 5T(\mathbf{e}_2) + T(\mathbf{e}_3) \\
= 2[3, 1, 2] - 5[2, -1, 4] + [6, 0, 1] \\
= [6 - 10 + 6, 2 + 5 + 0, 4 - 20 + 1] \\
= [2, 7, -15]
\]

3. Find the standard matrix representations of the following linear transformations.

(a) \( T([x_1, x_2]) = [x_1 + x_2, x_1 - 3x_2] \)

• The standard matrix representations are computed by computing the action of the linear transformation \( T \) on the standard basis vectors, and then using results as the columns of the corresponding matrix. For the case at hand we have

\[
\mathbf{e}_1 = [1, 0] \Rightarrow T(\mathbf{e}_1) = [1 + 0, 1 - 3(0)] = [1, 1] \\
\mathbf{e}_2 = [0, 1] \Rightarrow T(\mathbf{e}_2) = [0 + 1, 0 - 3(1)] = [1, -3]
\]

So the matrix corresponding to \( T \) is

\[
\begin{bmatrix}
1 & 1 \\
1 & -3
\end{bmatrix}
\]

(b) \( T([x_1, x_2, x_3]) = [x_1 + x_2 + x_3, x_1 + x_2, x_1] \)
• We proceed as in Part (a).

\[ \begin{align*}
e_1 &= [1, 0, 0] \Rightarrow T(e_1) = [1 + 0 + 0, 1 + 0] = [1, 1, 1] \\
e_2 &= [0, 1, 0] \Rightarrow T(e_2) = [0 + 1 + 0, 0 + 1] = [1, 1, 0] \\
e_3 &= [0, 0, 1] \Rightarrow T(e_3) = [0 + 0 + 1, 0 + 0] = [1, 0, 0]
\end{align*} \]

So the matrix corresponding to \( T \) is
\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]

\[\square\]

(c) \( T : \mathbb{R}^3 \to \mathbb{R}^2 : T([x_1, x_2, x_3]) = [x_1 + x_2 + x_3, 2x_1 + 2x_2 + 2x_3] \)

• We again compute the action of \( T \) on each standard basis vector \( e_i \) in the domain \( \mathbb{R}^3 \) of \( T \) and then use the results as the columns of \( A_T \):

\[ \begin{align*}
e_1 &= [1, 0, 0] \Rightarrow T(e_1) = [1 + 0 + 0, 2 + 0 + 0] = [1, 2] \\
e_2 &= [0, 1, 0] \Rightarrow T(e_2) = [0 + 1 + 0, 2 + 0 + 2] = [1, 2] \\
e_3 &= [0, 0, 1] \Rightarrow T(e_3) = [0 + 0 + 1, 0 + 2 + 0 + 2] = [1, 2]
\end{align*} \]

\[ A_T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \]

4. For each of the linear transformations \( T : \mathbb{R}^m \to \mathbb{R}^n \) in Problem 3, determine

\[ \text{Range} (T) := \{ y \in \mathbb{R}^n \mid y = T(x) \text{ for some } x \in \mathbb{R}^m \} \]

and

\[ \text{Kernel} (T) := \{ x \in \mathbb{R}^m \mid T(x) = 0_{\mathbb{R}^n} \} \]

• If the matrix corresponding to \( T \) is \( A_T \), then a basis for \( \text{Range} (T) \) coincides with a basis for the \( \text{ColSp} (A_T) \) and a basis for \( \text{Kernel} (T) \) coincides with a basis for \( \text{NullSp} (A_T) \). These two bases, in turn, can be calculated by reducing \( A_T \) to row echelon form and then interpreting that result accordingly. This I will do below for each of the matrices \( A_T \) computed in Problem 3.

(a) Here we found

\[ A_T = \begin{bmatrix} 1 & 1 & -3 \\ 1 & 0 & 1 \end{bmatrix} \]

This matrix row reduces to the following Reduced Row Echelon Form.

\[ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

Because there is a pivot in each column of the RREF, each column of \( A_T \) is a basis vector for \( \text{ColSp} (A_T) \approx \text{Range} (T) \). Thinking of \( \text{Range} (T) \) as a subspace of \( \mathbb{R}^2 \), and writing vectors in \( \mathbb{R}^2 \) horizontally, we have

\[ \text{ColSp} (A_T) = \text{span} \left( \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} \right) \Rightarrow \text{Range} (T) = \text{span} ([1, 1], [1, -3]) \]

Because there are no columns without pivots there are no free parameters in the solution of \( A_T x = 0 \). Therefore, \( x = 0 \) is the only solution and so \( \text{NullSp} (A_T) = \{0\} \). Thus,

\[ \text{Kernel} (T) = \text{NullSp} (A_T) = \{0\} \]

(b) Here we found

\[ A_T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \]
This matrix row reduces to the following matrix in RREF
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Just like in part (a), there is a pivot in every column of the RREF and so each column of the original matrix is a basis vector for $\text{ColSp} (A_T) \approx \text{Range} (T)$. Thus,

\[
\text{ColSp} (A_T) = \text{span} \left( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) \Rightarrow \text{Range} (T) = \text{span} ([1, 1, 1], [1, 1, 0], [1, 0, 0])
\]

Also, there are no free parameters in the solution set of $A_T x = 0$, and so

\[
\text{Kernel} (T) \approx \text{NullSp} (A_T) = \{0\}
\]

(c) Here we found

\[
A_T = \begin{bmatrix}
1 & 1 & 1 \\
2 & 2 & 2
\end{bmatrix}
\]

This matrix row reduces to the following RREF
\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

Since only the first column of the RREF contains a pivot, just the first column of $A_T$ will provide a basis for $\text{ColSp} (A_T) \approx \text{Range} (T)$. Thus,

\[
\text{ColSp} (A_T) = \text{span} \left( \begin{bmatrix} 1 \\ \end{bmatrix} \right) \Rightarrow \text{Range} (T) = \text{span} ([1, 2])
\]

There are two columns without pivots, thus two free parameters $x_2, x_3$ in the solution set of $A_T x = 0$. To get a basis for $\text{NullSp} (A_T) \approx \text{Kernel} (T)$, we’ll write down the general solution of $A_T x = 0$ and grab the basis vectors from that. From the RREF form of $A_T$ we get the following equations

\[
\begin{align*}
x_1 + x_2 + x_3 &= 0 \\
0 &= 0
\end{align*}
\]

\[
\Rightarrow \quad x_1 = -x_2 - x_3 \quad \Rightarrow \quad x = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}
\]

And so

\[
\text{NullSp} (A_T) = \text{span} \left( \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right) \Rightarrow \text{Kernel} (T) = \text{span} ([1, -1, 0], [-1, 0, 1])
\]

\[
x_1 + x_2 + x_3 = 0 \\
0 = 0
\]

5. If $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is defined by $T ([x_1, x_2]) = [2x_1 + x_2, x_1, x_1 - x_2]$ and $T' : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is defined by $T' ([x_1, x_2, x_3]) = [x_1 - x_2 + x_3, x_1 + x_2]$, find the standard matrix representation for the linear transformation $T' \circ T$ that carries $\mathbb{R}^2$ into $\mathbb{R}^2$. Find a formula for $(T' \circ T) ([x_1, x_2])$.

- The matrix representations corresponding to $T$ and $T'$ are

\[
M_T = \begin{bmatrix}
2 & 1 \\
1 & 0 \\
1 & -1
\end{bmatrix}, \quad M_{T'} = \begin{bmatrix}
1 & -1 & 1 \\
1 & 1 & 0 \\
1 & 1 & -1
\end{bmatrix}
\]

The matrix representation corresponding to $T' \circ T$ will be given by the product of the corresponding matrices

\[
M_{T' \circ T} = M_{T'} M_T = \begin{bmatrix}
1 & -1 & 1 \\
1 & 1 & 0 \\
1 & 1 & -1
\end{bmatrix} \begin{bmatrix}
2 & 1 \\
1 & 0 \\
1 & -1
\end{bmatrix} = \begin{bmatrix}
2 & 0 \\
3 & 1
\end{bmatrix}
\]
Hence

\[(T' \circ T) (x_1, x_2) = [2x_1, 3x_1 + x_2]\]