Math 3013
Solutions to Problem Set 5

1. Determine which of the following mappings are linear transformations.

(a) $T:\mathbb{R}^3\to\mathbb{R}^2 : T([x_1,x_2,x_3]) = [x_1+x_2, x_1-3x_2]$

- This mapping is linear since if $\mathbf{v}=[x_1,x_2,x_3]$

$$\begin{aligned}
T(\lambda\mathbf{v}) &= T(\lambda[x_1,x_2,x_3]) \\
&= T([\lambda x_1,\lambda x_2,\lambda x_3]) \\
&= [\lambda x_1+\lambda x_2, \lambda x_1 - 3\lambda x_2] \\
&= \lambda[x_1+x_2, x_1-3x_2] \\
&= \lambda T([x_1,x_2,x_3]) \\
&= \lambda T(\mathbf{v}) \quad (T \text{ preserves scalar multiplication})
\end{aligned}$$

and if $\mathbf{v}=[x_1,x_2,x_3]$ and $\mathbf{v}'=[x_1',x_2',x_3']$

$$\begin{aligned}
T(\mathbf{v}+\mathbf{v}') &= T([x_1+x_1', x_2+x_2', x_3+x_3']) \\
&= [x_1+x_1'+x_2+x_2', x_1+x_1'-3(x_2+x_2')] \\
&= [x_1+x_2, x_1-3x_2] + [x_1'+x_2', x_1'-3x_2'] \\
&= T(\mathbf{v})+T(\mathbf{v}') \quad (T \text{ preserves vector addition})
\end{aligned}$$

$\square$

(b) $T:\mathbb{R}^3\to\mathbb{R}^4 : T([x_1,x_2,x_3]) = [0,0,0,0]$

- This mapping is linear since if $\mathbf{v}=[x_1,x_2,x_3]$

$$\begin{aligned}
T(\lambda\mathbf{v}) &= T([\lambda x_1,\lambda x_2,\lambda x_3]) \\
&= [0,0,0,0] \\
&= \lambda[0,0,0,0] \\
&= \lambda T([x_1,x_2,x_3]) \\
&= \lambda T(\mathbf{v}) \quad (T \text{ preserves scalar multiplication})
\end{aligned}$$

and if $\mathbf{v}=[x_1,x_2,x_3]$ and $\mathbf{v}'=[x_1',x_2',x_3']$

$$\begin{aligned}
T(\mathbf{v}+\mathbf{v}') &= T([x_1+x_1', x_2+x_2', x_3+x_3']) \\
&= [0,0,0,0] \\
&= [0,0,0,0]+[0,0,0,0] \\
&= T(\mathbf{v})+T(\mathbf{v}') \quad (T \text{ preserves vector addition})
\end{aligned}$$

$\square$

(c) $T:\mathbb{R}^3\to\mathbb{R}^4 : T([x_1,x_2,x_3]) = [1,1,1,1]$

- This mapping is not linear since if $\mathbf{v}=[x_1,x_2,x_3]$

$$\begin{aligned}
T(\mathbf{v}) &= [1,1,1,1] \\
T(2\mathbf{v}) &= [1,1,1,1] \neq 2[1,1,1,1] = 2T(\mathbf{v})
\end{aligned}$$

So the mapping does not preserve scalar multiplication. $\square$

(d) $T:\mathbb{R}^2\to\mathbb{R}^3 : T([x_1,x_2]) = [x_1-x_2, x_2+1, 3x_1-2x_2]$

- This mapping is not linear since, e.g., if $\mathbf{v} = [1, 1]$
$$\begin{aligned} T(\mathbf{v}) &= [0, 2, 1] \\ T(2\mathbf{v}) &= T([2, 2]) = [0, 3, 2] \neq [0, 4, 2] = 2T(\mathbf{v}) \end{aligned}$$
So the mapping does not preserve scalar multiplication. $\square$

2. For each of the following, assume T is a linear transformation, from the data given, compute the specified value.

(a) Given $T([1, 0]) = [3, -1]$, and $T([0, 1]) = [-2, 5]$, find $T([4, -6])$.

- Because linear transformations preserve scalar multiplication and vector addition, they also preserve linear combinations:
$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)$$
Now take $\mathbf{e}_1 = [1, 0]$ and $\mathbf{e}_2 = [0, 1]$. Then
$$\begin{aligned} T([4, -6]) &= T(4\mathbf{e}_1 - 6\mathbf{e}_2) \\ &= 4T(\mathbf{e}_1) - 6T(\mathbf{e}_2) \\ &= 4[3, -1] - 6[-2, 5] \\ &= [12 + 12, -4 - 30] \\ &= [24, -34] \end{aligned}$$
$\square$

(b) Given $T([1, 0, 0]) = [3, 1, 2]$, $T([0, 1, 0]) = [2, -1, 4]$, and $T([0, 0, 1]) = [6, 0, 1]$, find $T([2, -5, 1])$.

- As in Part (a), we set $\mathbf{e}_1 = [1, 0, 0]$, $\mathbf{e}_2 = [0, 1, 0]$, and $\mathbf{e}_3 = [0, 0, 1]$ and then compute
$$\begin{aligned} T([2, -5, 1]) &= T(2\mathbf{e}_1 - 5\mathbf{e}_2 + \mathbf{e}_3) \\ &= 2T(\mathbf{e}_1) - 5T(\mathbf{e}_2) + T(\mathbf{e}_3) \\ &= 2[3, 1, 2] - 5[2, -1, 4] + [6, 0, 1] \\ &= [6 - 10 + 6, 2 + 5 + 0, 4 - 20 + 1] \\ &= [2, 7, -15] \end{aligned}$$
$\square$

3. Find the standard matrix representations of the following linear transformations.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2 : T([x_1, x_2]) = [x_1 + x_{2T}, x_1 - 3x_2]$

- The standard matrix representations are computed by computing the action of the linear transformation T on the standard basis vectors, and then using results as the columns of the corresponding matrix. For the case at hand we have
$$\begin{aligned} \mathbf{e}_1 &= [1, 0] \quad \Rightarrow \quad T(\mathbf{e}_1) = [1 + 0, 1 - 3(0)] = [1, 1] \\ \mathbf{e}_2 &= [0, 1] \quad \Rightarrow \quad T(\mathbf{e}_2) = [0 + 1, 0 - 3(1)] = [1, -3] \end{aligned}$$
So the matrix corresponding to T is
$$\begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}$$
$\square$

(b) $T : \mathbb{R}^3 \to \mathbb{R}^2 : T([x_1, x_2, x_3]) = [x_1 + x_2 + x_3, x_1 + x_2, x_1]$

- We proceed as in Part (a).

$$\begin{aligned}\mathbf{e}_1 &= [1,0,0] &\Rightarrow\quad & T(\mathbf{e}_1) = [1+0+0, 1+0, 1] = [1,1,1]\\ \mathbf{e}_2 &= [0,1,0] &\Rightarrow\quad & T(\mathbf{e}_2) = [0+1+0, 0+1, 0] = [1,1,0]\\ \mathbf{e}_3 &= [0,0,1] &\Rightarrow\quad & T(\mathbf{e}_3) = [0+0+1, 0+0, 0] = [1,0,0]\end{aligned}$$

So the matrix corresponding to T is

$$\left[\begin{array}{ccc} 1 & 1 & 1\\ 1 & 1 & 0\\ 1 & 0 & 0\end{array}\right]$$

$\square$

(c) $T : \mathbb{R}^3 \to \mathbb{R}^2 : T([x_1, x_2, x_3]) = [x_1 + x_2 + x_3, 2x_1 + 2x_2 + 2x_3]$

- Proceeding as in Part (a)

$$\mathbf{A}_T = \left[\begin{array}{ccc} 1 & 1 & 1\\ 2 & 2 & 2\end{array}\right]$$

4. For each of the linear transformations $T : \mathbb{R}^m \to \mathbb{R}^n$ in Problem 3, determine

$$Range(T) := \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y} = T(\mathbf{x}) \text{ for some } \mathbf{x} \in \mathbb{R}^m\}$$

and

$$Kernel(T) := \{\mathbf{x} \in \mathbb{R}^m \mid T(\mathbf{x}) = \mathbf{0}_{\mathbb{R}^n}\}$$

(a)

- We have

$$\begin{aligned} Range(T) &= ColSp(\mathbf{A}_T) = ColSp\left(\left[\begin{array}{cc} 1 & 1\\ 1 & -3\end{array}\right]\right)\\ Kernel(T) &= NullSp(\mathbf{A}_T) = \text{ solution set of } \mathbf{A}_T\mathbf{x} = \mathbf{0}\end{aligned}$$

$\mathbf{A}_T$ row reduces to the Reduced Row Echelon Form

$$\left[\begin{array}{cc} 1 & 0\\ 0 & 1\end{array}\right]$$

Since each column of this row echelon form contains a pivot, each column of $\mathbf{A}_T$ is a basis vector for the column space of $\mathbf{A}_T$. Thus,

$$Range(T) = ColSp(\mathbf{A}_T) = span\left(\left[\begin{array}{c} 1\\ 1\end{array}\right], \left[\begin{array}{c} 1\\ -3\end{array}\right]\right)$$

From the Reduced Row Echelon Form, we can also read of the solution set of $\mathbf{A}_T\mathbf{x} = \mathbf{0}$. We must have

$$\left.\begin{array}{l} x_1 = 0\\ x_2 = 0\end{array}\right\} \quad\Rightarrow\quad \mathbf{x} = \left[\begin{array}{c} 0\\ 0\end{array}\right]$$

Therefore,

$$Ker(T) = NullSp(\mathbf{A}_t) = \left\{\left[\begin{array}{c} 0\\ 0\end{array}\right]\right\}$$

(b)

- We proceed as in Part (a). The matrix $\mathbf{A}_T$ row reduces to a R.R.E.F.

$$\mathbf{A}_T = \left[\begin{array}{ccc} 1 & 1 & 1\\ 1 & 1 & 0\\ 1 & 0 & 0\end{array}\right] \to \left[\begin{array}{ccc} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right]$$

Each column of the R.R.E.F. contains a pivot, so each of the columns of $\mathbf{A}_T$ is a basis vector for the column space of $\mathbf{A}_T$ and

$$Range\,(T) = ColSp\,(\mathbf{A}_T) = span\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right)$$

The solution set of $\mathbf{A}_T\mathbf{x} = \mathbf{0}$ is

$$\left.\begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{array}\right\} \quad \Rightarrow \quad Ker\,(T) = NullSp\,(\mathbf{A}_T) = \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$$

(c)

- We proceed as in Part (a). The matrix $\mathbf{A}_T$ row reduces to a R.R.E.F.

$$\mathbf{A}_T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Only the first column of the R.R.E.F. contains a pivot and so

$$Range\,(T) = ColSp\,(\mathbf{A}_T) = span\left(\begin{bmatrix}1\\2\end{bmatrix}\right)$$

The solutions of $\mathbf{A}_T\mathbf{x} = \mathbf{0}$ can be read off the R.R.E.F. of $\mathbf{A}_T$:

$$\left.\begin{array}{r} x_1 + x_2 + x_3 = 0 \\ 0 = 0 \end{array}\right\} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix}-1\\1\\0\end{bmatrix} + x_3\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$$Ker\,(T) = NullSp\,(\mathbf{A}_T) = span\left(\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right)$$

5. If $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is defined by $T\,([x_1,x_2]) = [2x_1 + x_2, x_1, x_1 - x_2]$ and $T' : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is defined by $T'\,([x_1, x_2, x_3]) = [x_1 - x_2 + x_3, x_1 + x_2]$, find the standard matrix representation for the linear transformation $T' \circ T$ that carries $\mathbb{R}^2$ into $\mathbb{R}^2$. Find a formula for $(T' \circ T)\,([x_1, x_2])$.

- The matrix representations corresponding to T and T' are

$$\mathbf{M}_T = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} \quad , \quad \mathbf{M}_{T'} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

The matrix representation corresponding to $T' \circ T$ will be given by the product of the corresponding matrices

$$\mathbf{M}_{T' \circ T} = \mathbf{M}_{T'}\mathbf{M}_T = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$$

Hence

$$(T' \circ T)\,(x_1, x_2) = [2x_1, 3x_1 + x_2]$$

: $\square$

6. Determine whether the following statements are *true* or *false*.

(a) Every linear transformation is a function.

- True. $\square$

(b) Every function mapping $\mathbb{R}^n$ to $\mathbb{R}^m$ is a linear transformation.

- False. In order to be a linear transformation a function $f : \mathbb{R}^n \to \mathbb{R}^m$ must preserve scalar multiplication and vector addtion. $\square$

(c) Composition of linear transformations corresponds to multiplication of their standard matrix representations.

- True. $\square$

(d) Function composition is associative.

- True. $\square$

(e) An invertible linear transformation mapping $\mathbb{R}^n$ to itself has a unique inverse.

- True. (This follows from the corresponding theorem about invertible matrices.) $\square$

(f) The same matrix may be the standard matrix representation for several different linear transformations.

- False. (Unless one allows more general vector spaces - but idea won't be broached until Chapter 3.) $\square$

(g) A linear transformation having an $m \times n$ matrix as its standard matrix representation maps $\mathbb{R}^n$ into $\mathbb{R}^m$.

- True. $\square$

(h) If T and T' are different linear transformations mapping $\mathbb{R}^n$ into $\mathbb{R}^m$, then we may have $T(\mathbf{e}_i) = T'(\mathbf{e}_i)$ for all standard basis vectors $\mathbf{e}_i$ of $\mathbb{R}^n$.

- False. Linear transformations are determined uniquely by their standard matrix representations. $\square$

(i) If T and T' are different linear transformations mapping $\mathbb{R}^n$ into $\mathbb{R}^m$, then we may have $T(\mathbf{e}_i) = T'(\mathbf{e}_i)$ for some standard basis vectors $\mathbf{e}_i$ of $\mathbb{R}^n$.

- True. (So long as they are not all the same.) $\square$

(j) If $B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis for $\mathbb{R}^n$ and T and T' are linear transformations from $\mathbb{R}^n$ into $\mathbb{R}^m$, then $T(\mathbf{x}) = T'(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ if and only if $T(\mathbf{b}_i) = T'(\mathbf{b}_i)$ for $i = 1, 2, \ldots, n$.

- True. $\square$